SADDLE SOLUTIONS FOR A CLASS OF SYSTEMS OF PERIODIC AND REVERSIBLE SEMILINEAR ELLIPTIC EQUATIONS

FRANCESCA ALESSIO, PIERO MONTECCHIARI, ANDREA SFECCI

Abstract. We study systems of elliptic equations $-\Delta u(x)+F_u(x,u)=0$ with potentials $F\in C^2(\mathbb{R}^n,\mathbb{R}^m)$ which are periodic and even in all their variables. We show that if $F(x,u)$ has flip symmetry with respect to two of the components of x and if the minimal periodic solutions are not degenerate then the system has saddle type solutions on $\mathbb{R}^n$.

1. Introduction

We consider systems of semilinear elliptic equations

$$-\Delta u(x)+F_u(x,u)=0 \tag{PDE}$$

where

(F_1) $F\in C^2(\mathbb{R}^n\times\mathbb{R}^m;\mathbb{R})$ is 1-periodic in all its variables, $n,m\geq 1$.

When $n=1$ and $m\geq 1$, (PDE) are particular cases of the dynamical systems considered in the Aubry-Mather Theory ([9, 23, 24]). When $n>1$ and $m=1$ equations like (PDE) were studied by Moser in [25] (indeed in a much more general setting), and then by Bangert [13] and Rabinowitz and Stredulinsky [31], extending some of the results of the Aubry-Mather Theory for partial differential equations. These studies show the presence of a very rich structure of the set of minimal (or locally minimal) entire solutions of (PDE). In particular, when $m=1$ the set $\mathcal{M}_0$ of *minimal* periodic solutions of (PDE) is a non empty ordered set and if $\mathcal{M}_0$ is not a continuum then there exists another ordered family $\mathcal{M}_1$ of minimal entire solutions which are heteroclinic in one space variable to a couple of (extremal) periodic solutions $u<v$ (a gap pair in $\mathcal{M}_0$). If $\mathcal{M}_1$ is not a continuum the argument can be iterated to find more complex ordered classes of *minimal* heteroclinic type solutions and the process continues if the corresponding set of minimal heteroclinics contains gaps. Variational gluing arguments were then employed by Rabinowitz and Stredulinsky to construct various kinds of homoclinic, heteroclinic or more generally multitransition solutions as *local minima* of renormalized functionals associated to (PDE), see [31]. Other extensions of Moser's results, including changing slope or higher Morse index solutions, have been developed by Bessi [10, 11], Bolotin and Rabinowitz [12], de la Llave and Valdinoci [17, 33]. Recently, in a symmetric setting and correspondingly to the presence of a gap pair in $\mathcal{M}_0$ symmetric with respect to the origin, entire solutions of saddle type were found by Autuori, Alessio and Montecchiari in [2].

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All these results are based on the ordered structure of the set of minimal solutions of (PDE) in the case $m = 1$ and a key tool in their proofs is the Maximum Principle, which is no longer available when $m > 1$.

The study of (PDE) when $n, m > 1$ was initiated by Rabinowitz in [29, 30]. Denoting $L(u) = \frac{1}{2}|\nabla u|^2 + F(x, u)$ and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, periodic solutions to (PDE) were found as minima of the functional $J_0(u) = \int_{\mathbb{T}^n} L(u)dx$ on $E_0 = W^{1,2}(\mathbb{T}^n, \mathbb{R}^m)$ showing that

$$\mathcal{M}_0 = \{u \in E_0 \mid J_0(u) = c_0 := \inf_{E_0} J_0(u)\} \neq \varnothing.$$

Paul H. Rabinowitz studied the case of spatially reversible potentials F assuming

($\overline{F}_2$) F is even in x_i for $1 \leq i \leq n$

and proved in [29] that if $\mathcal{M}_0$ is constituted by isolated points then for each $v_- \in \mathcal{M}_0$ there is a $v_+ \in \mathcal{M}_0 \setminus \{v_-\}$ and a solution $u \in C^2(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m)$ of (PDE) that is heteroclinic in x_1 from v_- to v_+. These solutions were found by variational methods minimizing the renormalized functional

$$J(u) = \sum_{p\in\mathbb{Z}} J_{p,0}(u) := \sum_{p\in\mathbb{Z}} \left(\int_{T_{p,0}} L(u)\, dx - c_0 \right), \tag{1.1}$$

(where $T_{p,0} = [p, p+1] \times [0, 1]^{n-1}$) on the space

$$\Gamma(v_-, v_+) = \{u \in W^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m) \mid \|u - v_\pm\|_{L^2(T_{p,0},\mathbb{R}^m)} \to 0 \text{ as } p \to \pm\infty\}.$$

In [30] the existence of minimal double heteroclinics was obtained assuming that the elements of $\mathcal{M}_0$ are not degenerate critical points of J_0 and that the set $\mathcal{M}_1(v_-, v_+)$ of the minima of J on $\Gamma(v_-, v_+)$ is constituted by isolated points. This research line was continued by Montecchiari and Rabinowitz in [26] where, via variational methods, multitransition solutions of (PDE) were found by glueing different integer phase shifts of minimal heteroclinic connections.

The proof of these results do not use the ordering property of the solutions and adapts to the study of (PDE) some of the ideas developed to obtain multi-transition solutions for Hamiltonian systems (see e.g. [3], [28] and the references therein). Aim of the present paper is to show how these methods, in particular a refined study of the concentrating properties of the minimal heteroclinic solutions to (PDE), can be used in a symmetric setting to obtain saddle type solutions to (PDE).

Saddle solutions was first studied by Dang, Fife and Peletier in [16]. In that paper the authors considered Allen-Cahn equations $-\Delta u + W'(u) = 0$ on $\mathbb{R}^2$ with W an even double well potential. They proved the existence of a (unique) saddle solution $v \in C^2(\mathbb{R}^2)$ of that equation, i.e., a bounded entire solution having the same sign and symmetry of the product function x_1x_2 and being asymptotic to the minima of the potential W along any directions not parallel to the coordinate axes. The saddle solution can be seen as a phase transition with cross interface.

We refer to [14, 15, 6, 7, 27] for the study of saddle solutions in higher dimensions and to [1, 20, 8] for the case of systems of autonomous Allen-Cahn equations. Saddle solutions can be moreover viewed as particular k-end solutions (see [4, 18, 22, 19]).

In [5] the existence of saddle type solutions was studied for non autonomous Allen-Cahn type equations and this work motivated the paper [2] where solutions of saddle type for (PDE) were found in the case $m = 1$.

In the present paper we generalize the setting considered in [2] to the case $m > 1$. Indeed we consider to have potentials F satisfying (F_1) and the symmetry properties

(F_2) F is even in all its variables;

(F_3) F has flip symmetry with respect to the first two variables, i.e.,

$$F(x_1, x_2, x_3, ..., x_n, u) = F(x_2, x_1, x_3, ..., x_n, u) \text{ on } \mathbb{R}^n \times \mathbb{R}^m.$$

By [29] the set $\mathcal{M}_0$ of minimal periodic solution of (PDE) is not empty. The symmetry of F implies that any $v \in \mathcal{M}_0$ has components whose sign is constant on $\mathbb{R}^n$ and if $v \in \mathcal{M}_0$ then $(\nu_1 v_1, \dots, \nu_m v_m) \in \mathcal{M}_0$ for every $(\nu_1, \dots, \nu_m) \in \{\pm 1\}^m$ (see Lemma 2.2 below). In this sense we can say that $\mathcal{M}_0$ is symmetric with respect to the constant function $v_0 \equiv 0$.

As recalled above, in [2], where $m = 1$, a saddle solution was found when $\mathcal{M}_0$ has a gap pair symmetric with respect to the origin. In the case $m > 1$ we generalize this gap condition asking that $0 \notin \mathcal{M}_0$ and, following [30], we look for saddle solutions of (PDE) when any $v \in \mathcal{M}_0$ is not degenerate for J_0. We then assume

(N) $0 \notin \mathcal{M}_0$ and there exists $\alpha_0 > 0$ such that

$$J_0''(v)h \cdot h = \int_{[0,1]^n} |\nabla h|^2 + F_{u,u}(x, v(x))h \cdot h \, dx \geq \alpha_0 \|h\|^2_{L^2([0,1]^n, \mathbb{R}^m)}$$

for every $h \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ and every $v \in \mathcal{M}_0$.

The assumption (N) and the symmetries of F allow us to find heteroclinic connections between elements of $\mathcal{M}_0$ which are odd in the variable x_1. More precisely for $v \in \mathcal{M}_0$ these solutions are searched as minima of the functional J (see (1.1)) on the space

$$\Gamma(v) = \{u \in W^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m) \mid u \text{ is odd in } x_1, \lim_{p \to +\infty} \|u - v\|_{L^2([p,p+1] \times \mathbb{T}^{n-1}, \mathbb{R}^m)} = 0\}.$$

In §4, setting

$$c(v) = \inf_{u \in \Gamma(v)} J(u) \text{ for } v \in \mathcal{M}_0$$

we show that the set $\mathcal{M}_0^{min}$ of the elements $v_0 \in \mathcal{M}_0$ for which $c(v_0) = \min_{v \in \mathcal{M}_0} c(v)$ is not empty and such that if $v_0 \in \mathcal{M}_0^{min}$ then the set

$$\mathcal{M}(v_0) = \{u \in \Gamma(v_0) \mid J(u) = c(v_0)\}$$

is not empty and compact with respect to the $W^{1,2}(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^n)$ metric. The elements $u \in \mathcal{M}(v_0)$ are classical solutions to (PDE), odd in x_1, even and 1-periodic in $x_2, ..., x_n$ and satisfy the asymptotic condition

$$\|u - v_0\|_{W^{1,2}([p,p+1] \times \mathbb{T}^{n-1}, \mathbb{R}^m)} \to 0 \text{ as } p \to +\infty.$$

Our main result can now be stated as follows

Theorem 1.1. *Assume (F_1), (F_2), (F_3) and (N). Then, there exists a classical solution w of* (PDE) *such that every component w_i (for $i = 1 \dots, m$) satisfies*

(i) $w_i \geq 0$ *for* $x_1 x_2 > 0$*;*

(ii) w_i *is odd in* x_1 *and* x_2*, 1-periodic in* $x_3, ..., x_n$*;*

(iii) $w_i(x_1, x_2, x_3, ..., x_n) = w_i(x_2, x_1, x_3, ..., x_n)$ *in* $\mathbb{R}^n$.

Moreover there exists $v_0 \in \mathcal{M}_0^{min}$ *such that the solution* w *satisfies the asymptotic condition*

$$\mathrm{dist}_{W^{1,2}(\mathcal{R}_k,\mathbb{R}^m)}(w, \mathcal{M}(v_0)) \to 0, \quad as \quad k \to +\infty, \tag{1.2}$$

where $\mathcal{R}_k = [-k, k] \times [k, k+1] \times [0, 1]^{n-2}$.

Note that by (i) and (ii) any component of w has the same sign as the product function x_1x_2. Moreover by (1.2), since w is asymptotic as $x_2 \to +\infty$ to the compact set $\mathcal{M}(v_0)$ of odd heteroclinic type solutions, the symmetry of w implies that w is asymptotic to v_0 or $-v_0$ along any direction not parallel to the planes $x_1 = 0$, $x_2 = 0$. In this sense w is a saddle solution, representing a multiple transition between the *pure phases* v_0 and $-v_0$ with cross interface.

The proof of Theorem 1.1 uses a variational approach similar to the one already used in previous papers like [5, 2]. To adapt this approach to the case $m > 1$ and so to avoid the use of the Maximum Principle we need a refined analysis of the concentrating properties of the minimizing sequences. For that a series of preliminaries results is given in §2, §3, §4 while the proof of Theorem 1.1 is developed in §5.

2. The periodic solutions

In this section we recall some results obtained by Rabinowitz in [29], on minimal periodic solutions to (PDE). Moreover, following the argument in [2], we study some symmetry properties related to the assumptions (F_2) and (F_3). Here and in the following we will work under the not restrictive assumption

(F_4) $F \geq 0$ on $\mathbb{R}^n \times \mathbb{R}^m$.

Let us introduce the set

$$E_0 = W^{1,2}(\mathbb{T}^n, \mathbb{R}^m) = \{u \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^m) \mid u \text{ is 1-periodic in all its variables}\}$$

with the norm

$$\|u\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)} = \left(\sum_{i=1}^{m} \int_{[0,1]^n} (|\nabla u_i|^2 + |u_i|^2) dx \right)^{\frac{1}{2}}.$$

We define the functional $J_0 : E_0 \to \mathbb{R}$ as

$$J_0(u) = \int_{[0,1]^n} \tfrac{1}{2}|\nabla u|^2 + F(x, u)\, dx = \int_{[0,1]^n} L(u)\, dx\,. \tag{2.1}$$

and consider the minimizing set

$$\mathcal{M}_0 = \{u \in E_0 \mid J_0(u) = c_0\} \text{ where } c_0 = \inf_{u \in E_0} J_0(u)$$

Then in [29], [30] it is shown

Lemma 2.1. *Assume* (F_1), *then* $\mathcal{M}_0 \neq \varnothing$. *Moreover, setting* $[u] = \int_{[0,1]^n} u\, dx$, *we have that*

1. $\hat{\mathcal{M}}_0 = \{u \in \mathcal{M}_0 \mid [u] \in [0, 1]^m\}$ *is a compact set in* E_0*;*
2. *if* $(u_k)_k \subset E_0$, *with* $[u_k] \in [0, 1]^m$, *is a minimizing sequence for* J_0, *then there exists* $u \in \hat{\mathcal{M}}_0$ *such that* $u_k \to u$ *in* E_0 *up to subsequences;*

(3) *For every* $\rho > 0$ *there exists* $\beta(\rho) > 0$ *such that if* $u \in E_0$ *is such that*

$$\mathrm{dist}_{W^{1,2}([0,1]^n,\mathbb{R}^m)}(u, \mathcal{M}_0) := \inf_{v \in \mathcal{M}_0} \|u - v\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)} > \rho,$$

then $J_0(u) - c_0 \geq \beta(\rho)$;

(4) *If* (F_2) *holds, any* $u \in \mathcal{M}_0$ *minimizes also* $I(u) = \int_{[0,\frac{1}{2}]^n} L(u)\,dx$ *on* $W^{1,2}([0,\frac{1}{2}]^n, \mathbb{R}^m)$. *As a consequence, every* $u \in \mathcal{M}_0$ *is symmetric in* x_i *about* $x_i = 0$ *and* $x_i = \frac{1}{2}$ *for every index* i *and* u *is even in* x_i *for every index* i;

(5) *If* (F_2) *holds, there results* $c_0 = \inf_{u \in W^{1,2}([0,1]^n,\mathbb{R}^m)} J_0(u)$. *Furthermore, if* $u \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ *verifies* $J_0(u) = c_0$, *then for every* $i = 1, 2, ..., n$, u *is symmetric in* x_i *about* $x_i = \frac{1}{2}$ *and hence* $u \in \mathcal{M}_0$.

Assumption (F_2), in particular the even parity of F with respect to the components of u, provides that the elements in $\mathcal{M}_0$ have components with definite sign, thanks to the unique extension property (see [29], Proposition 3).

Lemma 2.2. *Assume* (F_1), (F_2) *and* $0 \notin \mathcal{M}_0$. *If* $u = (u_1, \ldots, u_m) \in \mathcal{M}_0$ *then, for every* $i = 1, ..., m$, *one has either* $u_i \geq 0$, *or* $u_i \leq 0$ *on* $[0,1]^n$ *and* u *does not vanish on open sets. Moreover,* $(\nu_1 u_1, \ldots, \nu_m u_m) \in \mathcal{M}_0$ *for every* $(\nu_1, \ldots, \nu_m) \in \{\pm 1\}^m$.

Proof. It is sufficient to observe that if $u = (u_1, \ldots, u_m) \in \mathcal{M}_0$ then, since F is even with respect to the components of u, we have

i) $\bar{u} = (|u_1|, \ldots, |u_m|) \in \mathcal{M}_0$ and
ii) $(\nu_1 u_1, \ldots, \nu_m u_m) \in \mathcal{M}_0$ for every $(\nu_1, \ldots, \nu_m) \in \{\pm 1\}^m$.

Property (ii) gives the second part of the statement while by (i) and the unique extension property proved in [29], we obtain that the components of u do not change sign. If u vanishes on an open set, the unique continuation property gives $u \equiv 0$, giving a contradiction and concluding the proof. □

On the other hand, assumption (F_3) gives more structure on the set $\mathcal{M}_0$: its elements have a flip symmetry property. Indeed, setting $T^+ = \{x \in [0,1]^n \mid x_1 \leq x_2\}$, for every $u \in W^{1,2}(T^+, \mathbb{R}^m)$, let us define $\tilde{u} \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ as

$$\tilde{u}(x) = \begin{cases} u(x), & x \in T^+, \\ u(x_2, x_1, x_3, \ldots, x_n), & x \in [0,1]^n \setminus T^+ . \end{cases} \tag{2.2}$$

Then, we have

Lemma 2.3. *If* $u \in \mathcal{M}_0$ *then,* $u \equiv \tilde{u}$ *in* $[0,1]^n$.

Proof. Given $u \in \mathcal{M}_0$, without loss of generality, we assume

$$\int_{T^+} L(u)\,dx \leq \int_{[0,1]^n \setminus T^+} L(u)\,dx\,.$$

Since $\tilde{u} \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ by Lemma 2.1-(5) we have $J_0(\tilde{u}) \geq c_0$. By the previous inequality we get

$$c_0 = J_0(u) = \int_{T^+} L(u)\,dx + \int_{[0,1]^n \setminus T^+} L(u)\,dx \geq 2 \int_{T^+} L(u)\,dx = J_0(\tilde{u}) \geq c_0.$$

Hence, again by Lemma 2.1-(5), $\tilde u \in \mathcal{M}_0$. By the unique extension property of the solutions of (PDE) (cf. [29], Proposition 3), we have $\tilde u \equiv u$ in $[0,1]^n$. □

As an immediate consequence, using Lemma 2.1-(5), we have the following.

Lemma 2.4. *There results*

$$\min_{u\in W^{1,2}(T^+,\mathbb{R}^m)} \int_{T^+} L(u)\,dx = \tfrac{c_0}{2}. \tag{2.3}$$

Moreover, if $u \in W^{1,2}(T^+,\mathbb{R}^m)$ *verifies* $\int_{T^+} L(u)\,dx = \frac{c_0}{2}$, *then* $\tilde u \in \mathcal{M}_0$.

Remark 2.5. Lemma 2.3 tells us that the elements of $\mathcal{M}_0$ are symmetric with respect to the diagonal iperplane $\{x \in \mathbb{R}^n \mid x_1 = x_2\}$ and by Lemma 2.4 they can be found by minimizing $\int_{T^+} L(v)\,dx$ on $W^{1,2}(T^+,\mathbb{R}^m)$. Analogously, setting $T^- = [0,1]^n \setminus T^+$, we can find the elements of $\mathcal{M}_0$ by minimizing $\int_{T^-} L(v)\,dx$ on $W^{1,2}(T^-,\mathbb{R}^m)$ or, by periodicity, by minimizing $\int_T L(v)\,dx$ on $W^{1,2}(T,\mathbb{R}^m)$ whenever $T = p + T^\pm$ with $p \in \mathbb{Z}^n$. For future references it is important to note that this property implies in particular that $u \in \mathcal{M}_0$ if and only if u is a minimizer of the functional $\int_{\sigma_0} L(v)\,dx$ on $W^{1,2}(\sigma_0,\mathbb{R}^m)$ where

$$\sigma_0 = \{x \in \mathbb{R}\times[0,1]^{n-1} \,|\, x_2 - 1 \le x_1 \le x_2\}.$$

More precisely $c_0 = \inf_{v\in W^{1,2}(\sigma_0,\mathbb{R}^m)} \int_{\sigma_0} L(v)\,dx$ and $u \in \mathcal{M}_0$ if and only if $\int_{\sigma_0} L(u)\,dx = c_0$. From Lemma 2.1-(3) we recover an analogous property in $W^{1,2}(\sigma_0,\mathbb{R}^m)$: for any $r > 0$ there exists $\beta(r) > 0$ such that if $u \in W^{1,2}(\sigma_0,\mathbb{R}^m)$ verifies $\int_{\sigma_0} L(u)\,dx \le c_0 + \beta(r)$, then $\mathrm{dist}_{W^{1,2}(\sigma_0,\mathbb{R}^m)}(u,\mathcal{M}_0) \le r$.

Note that by Lemma 2.1-(1) and the assumption (N) we plainly derive that

(N_0) $\hat{\mathcal{M}}_0$ is a finite set and $0 \notin \hat{\mathcal{M}}_0$,

where we recall that $\hat{\mathcal{M}}_0 = \{u \in \mathcal{M}_0 \mid [u] \in [0,1]^m\}$ and note that $\mathcal{M}_0 = \hat{\mathcal{M}}_0 + \mathbb{Z}^m$.

Note finally that by (N_0), setting

$$r_0 := \min\left\{\|u - v\|_{L^2(\mathbb{T}^n,\mathbb{R}^m)} \mid u, v \in \mathcal{M}_0\,, u \not\equiv v\right\}, \tag{2.4}$$

we have $r_0 > 0$.

3. The variational setting for Heteroclinic connections

This section is devoted to introduce the variational framework to study solutions of (PDE) which are heteroclinic between minimal periodic solutions. We follow some arguments in [29], [26], introducing the renormalized functional J and studying some of its basic properties.

Let us define the set

$$E = \{u \in W^{1,2}_{loc}(\mathbb{R}^n,\mathbb{R}^m) \mid u \text{ is 1-periodic in } x_2,\dots x_n\}.$$

For any $u \in E$ we consider the functional

$$J(u) = \sum_{p\in\mathbb{Z}} J_{p,0}(u),$$

where, denoting $T_{p,0} = [p, p+1] \times [0,1]^{n-1}$,

$$J_{p,0}(u) = \int_{T_{p,0}} L(u)\, dx - c_0, \quad \forall p \in \mathbb{Z}.$$

Denoting briefly $u(\cdot + p)$ the shifting of the function u with respect to the first coordinate (that is, $u(\cdot + p) = u(\cdot + p\boldsymbol{e}_1)$ where $\boldsymbol{e}_1 = (1, 0, ..., 0)$), note that by periodicity we have

$$J_{p,0}(u) = \int_{[0,1]^n} L(u(\cdot + p))\, dx - c_0 = J_0(u(\cdot + p)) - c_0, \quad \forall p \in \mathbb{Z}.$$

Then, by Lemma 2.1, we have $J_{p,0}(u) \geq 0$ for any $u \in E$ and $p \in \mathbb{Z}$, from which J is non-negative on E.

Lemma 3.1. *The functional $J : E \to \mathbb{R}$ is weakly lower semicontinuous.*

Proof. Consider a sequence $(u_k)_k$ such that $u_k \to u$ weakly in E. Then, for every $\ell \in \mathbb{N}$, by the weak lower semicontinuity of J_0, and hence of $J_{p,0}$, we have $\sum_{p=-\ell}^{\ell} J_{p,0}(u) \leq \sum_{p=-\ell}^{\ell} J_{p,0}(u_k)$. If $J(u) = +\infty$, then we obtain easily $\liminf_k J(u_k) = +\infty$. So, let us assume $J(u) < +\infty$, then for any $\varepsilon > 0$ we have that there exists $\ell \in \mathbb{N}$ such that $\sum_{|p|>\ell} J_{p,0}(u) < \varepsilon$. We get

$$\liminf_k J(u_k) \geq \liminf_k \sum_{p=-\ell}^{\ell} J_{p,0}(u_k) \geq \sum_{p=-\ell}^{\ell} J_{p,0}(u) > J(u) - \varepsilon\,,$$

thus finishing the proof. □

Using the notation introduced above, note that if $u \in E$ is such that $J(u) < +\infty$, then $J_{p,0}(u) \to 0$ as $|p| \to +\infty$, that is, the sequence $(u(\cdot + p))_{p\in\mathbb{Z}}$ is such that $J_0(u(\cdot + p)) \to c_0$ as $p \to \pm\infty$. Hence, by Lemma 2.1-(3), there exist $u_\pm \in \mathcal{M}_0$ such that, up to a subsequence, $u(\cdot + p) \to u_\pm$ as $p \to \pm\infty$ in E_0. Using this remark and the local compactness of $\mathcal{M}_0$ given by (N_0), we are going to prove some *concentration properties* of the minimizing sequence of the functional J.

First of all, let us consider the functional $J_{p,0} + J_{p+1,0}$ for a certain fixed integer p. Notice that, by Lemma 2.1-(5),

$$\min_{u\in E} J_{p,0}(u) + J_{p+1,0}(u) = 0$$

and the set of minima coincide with $\mathcal{M}_0$. We introduce the following distance

$$\mathrm{dist}_p(u, A) = \inf\{\|u - v\|_{W^{1,2}(T_{p,0}\cup T_{p+1,0},\mathbb{R}^m)} \mid v \in A\}\,.$$

Remark 3.2. Let us fix some constants that will be used in rest of the paper. By Lemma 2.1-(3), we have that for any $r > 0$ there exists $\lambda(r) > 0$ such that

(3.1) if $u \in E$ satisfies $J_{p,0}(u) + J_{p+1,0}(u) \leq \lambda(r)$ for a $p \in \mathbb{Z}$, then $\mathrm{dist}_p(u, \mathcal{M}_0) \leq r$.

It is not restrictive to assume that the function with $r \mapsto \lambda(r)$ is non-decreasing.
On the other hand for every $\lambda > 0$ if we set

$$\rho(\lambda) = \sup\{\mathrm{dist}_p(u, \mathcal{M}_0) \mid u \in E \text{ with } J_{p,0}(u) + J_{p+1,0}(u) \leq \lambda,\, p \in \mathbb{Z}\}$$

we get $\rho(\lambda) > 0$ and that $\lambda \mapsto \rho(\lambda)$ is non-decreasing. Moreover, for every $\varepsilon > 0$, since if $J_{p,0}(u) + J_{p+1,0}(u) \le \lambda(\varepsilon)$ for a certain $p \in \mathbb{Z}$, then $\mathrm{dist}_p(u, \mathcal{M}_0) \le \varepsilon$, we obtain $\rho(\lambda) \le \rho(\lambda(\varepsilon)) \le \varepsilon$ for every $\lambda \in (0, \lambda(\varepsilon)]$, so that $\lim_{\lambda \to 0^+} \rho(\lambda) = 0$ holds. Hence, recalling the definition of r_0 in (2.4), we can fix $\lambda_0 > 0$ satisfying $\rho(\lambda_0) \le \frac{r_0}{4}$.
Finally, we can define

$$\Lambda(r) = \sup\{J_{p,0}(u) \mid u \in E \text{ and } p \in \mathbb{Z} \text{ are such that } \mathrm{dist}_p(u, \mathcal{M}_0) \le 2r\} \tag{3.2}$$

which is non-decreasing and $\lim_{r \to 0} \Lambda(r) = 0$. Then we fix $r_1 \in (0, \frac{r_0}{4})$ such that $\Lambda(r) \le \frac{\lambda_0}{8}$ for every $r \in (0, r_1]$.

We say that a set $\mathcal{I} \subseteq \mathbb{Z}$ is a set of consecutive integers if it is of the form $\{\ell \in \mathbb{Z} \,|\, p \le \ell < p + k\}$ or $\{\ell \in \mathbb{Z} \,|\, p - k < \ell \le p\}$ for a $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{+\infty\}$. If $u \in E$ is such that $J_{p,0}$ is small enough for some consecutive integers $p \in \mathcal{I}$, then, using (N_0), we can prove that, in the corresponding sets $T_{p,0}$, u is "near" to an element of $\mathcal{M}_0$, the same for all $p \in \mathcal{I}$. Indeed we have

Lemma 3.3. *Given $\lambda \in (0, \frac{\lambda_0}{2}]$, $u \in E$ and a set of consecutive integers $\mathcal{I}$, if $J_{p,0}(u) \le \lambda$ for any $p \in \mathcal{I}$, then there exists $v \in \mathcal{M}_0$ such that $\|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \le \rho(2\lambda) \le \frac{r_0}{4}$, for every $p \in \mathcal{I}$.*

Proof. Let $p \in \mathcal{I}$ be such that $p + 1 \in \mathcal{I}$. Then $J_{p,0}(u) + J_{p+1,0}(u) \le 2\lambda \le \lambda_0$ and, by Remark 3.2 and the definition of λ_0, $\mathrm{dist}_p(u, \mathcal{M}_0) \le \rho(2\lambda) \le \rho(\lambda_0) \le \frac{r_0}{4}$. Then, by (N_0) and the choice of r_0 in (2.4), we can find $v_p \in \mathcal{M}_0$ such that

$$\|u - v_p\|_{W^{1,2}(T_{p,0} \cup T_{p+1,0},\mathbb{R}^m)} \le \frac{r_0}{4}$$

from which $\|u - v_p\|_{W^{1,2}(T_{k,0},\mathbb{R}^m)} \le \frac{r_0}{4}$ for $k = p, p + 1$. If $p + 2 \in \mathcal{I}$, repeating the argument with the couple of indices $p + 1$ and $p + 2$ we find $v_{p+1} \in \mathcal{M}_0$ such that $\|u - v_{p+1}\|_{W^{1,2}(T_{k,0},\mathbb{R}^m)} \le \frac{r_0}{4}$ for $k = p + 1, p + 2$. By the choice of r_0 in (2.4), we conclude that $v_{p+1} = v_p$ and the lemma follows. □

Moreover, using the notations introduced above, we have

Lemma 3.4. *If $u \in W^{1,2}(T_{p,0} \cup T_{p+1,0}, \mathbb{R}^m)$ then*

$$\|u(\cdot + p) - u(\cdot + (p + 1))\|^2_{L^2([0,1]^n,\mathbb{R}^m)} \le 2(J_{p,0}(u) + J_{p+1,0}(u) + 2c_0).$$

Proof. Setting $y = (x_2, \dots, x_n)$, we have

$$\|u(\cdot + p) - u(\cdot + (p + 1))\|^2_{L^2([0,1]^n,\mathbb{R}^m)} = \int_p^{p+1} \int_{[0,1]^{n-1}} |u(x_1 + 1, y) - u(x_1, y)|^2 dy\, dx_1$$

and so there exists $\bar{x}_1 \in (p, p + 1)$ such that

$$\int_{[0,1]^{n-1}} |u(\bar{x}_1 + 1, y) - u(\bar{x}_1, y)|^2 dy \ge \|u(\cdot + p) - u(\cdot + (p + 1))\|^2_{L^2([0,1]^n,\mathbb{R}^m)}.$$

On the other hand, by Hölder inequality,

$$\begin{aligned}2(J_{p,0}(u)+J_{p+1,0}(u)+2c_0) &\geq \int_p^{p+2}\int_{[0,1]^{n-1}} |\partial_{x_1}u(x_1,y)|^2 dy\,dx_1\\ &\geq \int_{[0,1]^{n-1}}\int_{\bar{x}_1}^{\bar{x}_1+1} |\partial_{x_1}u(x_1,y)|^2 dx_1\,dy\\ &\geq \int_{[0,1]^{n-1}} |u(\bar{x}_1+1,y)-u(\bar{x}_1)|^2 dy\\ &\geq \|u(\cdot+p)-u(\cdot+(p+1))\|^2_{L^2([0,1]^n,\mathbb{R}^m)}\end{aligned}$$

completing the proof. □

By the previous lemmas we obtain that the elements in the sublevels of J satisfy the following boundeness property.

Lemma 3.5. *For every $\Lambda > 0$ there exists $R > 0$ such that for every $u \in E$ satisfying $J(u) \leq \Lambda$ one has $\|u(\cdot+p)-u(\cdot+q)\|_{L^2([0,1]^n,\mathbb{R}^m)} \leq R$ for any $p, q \in \mathbb{Z}$.*

Proof. Let $u \in E$ be such that $J(u) \leq \Lambda$. We define $\mathcal{J}(u) = \{k \in \mathbb{Z} \mid J_{k,0}(u) \geq \frac{\lambda_0}{2}\}$ and note that the number $l(u)$ of elements of $\mathcal{J}(u)$ is at most $[\frac{2\Lambda}{\lambda_0}]+1$, where $[\cdot]$ denotes the integer part. Then, the set $\mathbb{Z}\setminus\mathcal{J}(u)$ is constituted by $\bar{l}(u)$ sets of consecutive elements of $\mathbb{Z}$, $\mathcal{I}_i(u)$, with $\bar{l}(u) \leq l(u)+1$. By the triangular inequality, for any $p, q \in \mathbb{Z}$, we obtain

$$\begin{aligned}\|u(\cdot+p)-u(\cdot+q)\|_{L^2([0,1]^n,\mathbb{R}^m)} &\leq l(u)\sup_{k\in\mathcal{J}(u)}\|u(\cdot+k)-u(\cdot+k+1)\|_{L^2([0,1]^n,\mathbb{R}^m)}\\ &\quad+\sum_{i=1}^{\bar{l}(u)}\sup_{p,q\in\mathcal{I}_i(u)}\|u(\cdot+p)-u(\cdot+q)\|_{L^2([0,1]^n,\mathbb{R}^m)}\\ &\leq l(u)(2(\Lambda+2c_0))^{\frac{1}{2}}+\bar{l}(u)\tfrac{r_0}{2}. \qquad (3.3)\end{aligned}$$

where the first term in (3.3) follows by the application of Lemma 3.4, since

$$2(J_{k,0}(u)+J_{k+1,0}(u)+2c_0) \leq 2(J(u)+2c_0) \leq 2(\Lambda+2c_0), \quad \forall k \in \mathbb{Z},$$

while the second one follows by the definition of $\mathcal{I}_i(u)$ and Lemma 3.3.
Since $\bar{l}(u) \leq l(u)+1$ and $l(u) \leq [\frac{2c}{\lambda_0}]+1$, the lemma follows by choosing $R = ([\frac{2\Lambda}{\lambda_0}]+1)(2(\Lambda+2c_0))^{\frac{1}{2}}+([\frac{2\Lambda}{\lambda_0}]+2)\frac{r_0}{2}$. □

The following lemma states the weak compactness of the sublevels of the functional J.

Lemma 3.6. *Given any $\Lambda > 0$, let $(u_k)_k \subset E$ be a sequence such that $J(u_k) \leq \Lambda$ for every $k \in \mathbb{N}$ and let $(p_k)_k$ be a sequence of integers. Assume that there exist $\bar{R} < +\infty$ and $v \in \mathcal{M}_0$ such that $\|u_k - v\|_{W^{1,2}(T_{p_k,0},\mathbb{R}^m)} \leq \bar{R}$ for all $k \in \mathbb{N}$. Then, there exists $u \in E$ with $J(u) \leq \Lambda$ such that, up to a subsequence, $u_k \to u$ weakly in E.*

Proof. First note that, by Lemma 3.5, there exists $R>0$ such that if $u\in E$ and $J(u)\le\Lambda$ then $\|u(\cdot+p)-u(\cdot+q)\|_{L^2([0,1]^n,\mathbb{R}^m)}\le R$ for any $p,q\in\mathbb{Z}$. If $\|u-v\|_{W^{1,2}(T_{\ell,0},\mathbb{R}^m)}\le\bar R$ for some $\ell\in\mathbb{Z}$ and $v\in\mathcal{M}_0$, by triangular inequality for any $p\in\mathbb{Z}$ we obtain

$$\begin{aligned}\|u-v\|_{L^2(T_{p,0},\mathbb{R}^m)}&=\|u(\cdot+p)-v\|_{L^2([0,1]^n,\mathbb{R}^m)}\\&\le\|u(\cdot+p)-u(\cdot+\ell)\|_{L^2([0,1]^n,\mathbb{R}^m)}+\|u(\cdot+\ell)-v\|_{L^2([0,1]^n,\mathbb{R}^m)}\le R+\bar R.\end{aligned}$$

Consider now a sequence as in the statement, setting $Q_L=[-L,L]\times[0,1]^{n-1}$ for $L\in\mathbb{N}$, we get

$$\|u_k-v\|^2_{L^2(Q_L,\mathbb{R}^m)}+\|\nabla u_k\|^2_{L^2(Q_L,\mathbb{R}^m)}\le 2L(\bar R+R)^2+4Lc_0+2\Lambda.$$

Hence, $(u_k-v)_k$ is bounded in $W^{1,2}(Q_L,\mathbb{R}^m)$ for any $L\in\mathbb{N}$ and, by a diagonal argument and the weak lower semicontinuity of J, the statement follows. ☐

By Lemma 3.3 we also deduce the following result concerning the asymptotic behaviour of the functions in the sublevels of J.

Lemma 3.7. *If $J(u)<+\infty$, there exist $v^\pm\in\mathcal{M}_0$ such that*

$$\|u-v^\pm\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\to 0\quad\textit{as}\quad p\to\pm\infty.$$

Proof. Since $J(u)<+\infty$, we have $J_{p,0}(u)\to 0$ as $|p|\to+\infty$ and there exists $\bar p$ such that $J_{p,0}(u)\le\frac{\lambda_0}{2}$ for any $|p|\ge\bar p$. Thus, by Lemma 3.3, there exists $v^\pm\in\mathcal{M}_0$ such that $\|u-v^+\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\le\frac{r_0}{4}$ for $p\ge\bar p$ and $\|u-v^-\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\le\frac{r_0}{4}$ for $p\le-\bar p$.
Hence the sequence $(u(\cdot+p))_{p\in\mathbb{N}}$ is such that $\|u(\cdot+p)-v^+\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)}\le\frac{r_0}{4}$ for every $p\ge\bar p$ and $J_0(u(\cdot+p))-c_0=J_{p,0}(u)\to 0$ as $p\to+\infty$. Then, by Lemma 2.1, $\|u-v^+\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}=\|u(\cdot+p)-v^+\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)}\to 0$ as $p\to+\infty$. Analogously we obtain that $\|u-v^-\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\to 0$ as $p\to-\infty$ ☐

By Lemma 3.7, if $u\in E$ satisfies $J(u)<+\infty$ we can view it as an heteroclinic or homoclinic connection between two periodic solutions v^- and v^+ belonging to $\mathcal{M}_0$. Hence, we can consider elements of E belonging to the classes

$$\Gamma(v^-,v^+)=\big\{u\in E\mid \|u-v^\pm\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\to 0\,\text{as}\,p\to\pm\infty\big\}$$

where $v^\pm\in\mathcal{M}_0$.

We note that by Lemma 3.6, every sequence $(u_k)_{k\in\mathbb{N}}\subset\Gamma(v^-,v^+)$ with $J(u_k)\le\Lambda$ for all $k\in\mathbb{N}$, admits a subsequence which converges weakly to some $u\in E$. Indeed, since $\|u_k-v^+\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\to 0$ as $p\to+\infty$ for every $k\in\mathbb{N}$, fixed $\bar R>0$ there exists $p_k\in\mathbb{N}$ such that $\|u_k-v^+\|_{W^{1,2}(T_{p_k,0},\mathbb{R}^m)}\le\bar R$ and since $J(u_k)\le\Lambda$, by Lemma 3.6, there exists $u\in E$ such that, up to a subsequence, $u_k\to u$ weakly as $k\to+\infty$.

In particular, given $v^\pm\in\mathcal{M}_0$ and setting

$$c(v^-,v^+)=\inf_{u\in\Gamma(v^-,v^+)}J(u)\,,$$

as in [29], we obtain that for any $v^-\in\mathcal{M}_0$ there exist $v^+\in\mathcal{M}_0\setminus\{v^-\}$ and $u\in\Gamma(v^-,v^+)$ such that $c(v^-,v^+)=J(u)$. Moreover, it can be proved that any $u\in\Gamma(v^-,v^+)$ such that $c(v^-,v^+)=J(u)$ is a classical solution of (PDE) (see Theorem 3.3 in [29]).

Finally, we have that $\inf_{v^-\not\equiv v^+}c(v^-,v^+)>0$ as a consequence of the following lemma.

Lemma 3.8. *For every $v^{\pm} \in \mathcal{M}_0$ with $v^- \not\equiv v^+$, we have $c(v^-, v^+) \geq \frac{\lambda_0}{2}$. Moreover, $c(v^-, v^+) \to +\infty$ as $\|v^+ - v^-\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)} \to +\infty$.*

Proof. Assume that there exists $u \in \Gamma(v^-, v^+)$ satisfying $J(u) < \frac{\lambda_0}{2}$. Then $J_{p,0}(u) < \frac{\lambda_0}{2}$ for every $p \in \mathbb{Z}$, so that by Lemma 3.3 there exists $v \in \mathcal{M}_0$ such that $\|u-v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq \frac{r_0}{4}$ for every $p \in \mathbb{Z}$. Since $u \in \Gamma(v^-, v^+)$ we know that $\|u-v^-\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \to 0$ as $p \to -\infty$ and $\|u - v^+\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \to 0$ as $p \to +\infty$, so that by (2.4) we would have $v^- = v = v^+$ giving a contradiction.
In order to prove the second part of the statement, assume the existence of two sequences $(v_k^-)_k$ and $(v_k^+)_k$ in $\mathcal{M}_0$ such that $(c(v_k^-, v_k^+))_k$ is bounded while $\|v_k^+ - v_k^-\|_{W^{1,2}(\mathbb{T}^n,\mathbb{R}^m)} \to +\infty$ as $k \to +\infty$. Since $(c(v_k^-, v_k^+))_k$ is bounded, we can find $\Lambda > 0$ and a sequence $(u_k)_k$, with $u_k \in \Gamma(v_k^-, v_k^+)$, such that $J(u_k) \leq \Lambda$, for every index k. Hence, by Lemma 3.5, there exists $R > 0$ such that $\|u_k(\cdot+p) - u_k(\cdot+q)\|_{L^2([0,1]^n,\mathbb{R}^m)} \leq R$ for every $k \in \mathbb{N}$ and $p, q \in \mathbb{Z}$. Moreover, for every $\varepsilon > 0$ and $k \in \mathbb{N}$, since $u_k \in \Gamma(v_k^-, v_k^+)$, there exist $p_k, q_k \in \mathbb{Z}$ such that $\|u_k - v_k^-\|_{W^{1,2}(T_{p_k,0},\mathbb{R}^m)} < \varepsilon$ and $\|u_k - v_k^+\|_{W^{1,2}(T_{q_k,0},\mathbb{R}^m)} < \varepsilon$ for every $k \in \mathbb{N}$. In particular we get

$$\begin{aligned}\|v_k^+ - v_k^-\|_{L^2([0,1]^n,\mathbb{R}^m)} \leq \|v_k^- &- u_k(\cdot + p_k)\|_{L^2([0,1]^n,\mathbb{R}^m)} \\ &+ \|u_k(\cdot + p_k) - u_k(\cdot + q_k)\|_{L^2([0,1]^n,\mathbb{R}^m)} \\ &+ \|v_k^+ - u_k(\cdot + q_k)\|_{L^2([0,1]^n,\mathbb{R}^m)} \leq \varepsilon + R + \varepsilon\end{aligned}$$

since, by periodicity, $\|v_k^{\pm} - u_k(\cdot + p)\|_{L^2([0,1]^n,\mathbb{R}^m)} = \|v_k^{\pm} - u_k\|_{W^{1,2}(L^2,\mathbb{R}^m)}$ for any $k \in \mathbb{N}$, $p \in \mathbb{Z}$. Finally, since $\|\nabla v\|^2_{L^2([0,1]^n,\mathbb{R}^m)} \leq 2c_0$ for every $v \in \mathcal{M}_0$, we recover $\|v^+ - v^-\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)} \leq 2\sqrt{2c_0} + 2\varepsilon + R$ in contradiction with $\|v_k^+ - v_k^-\|_{W^{1,2}(\mathbb{T}^n,\mathbb{R}^m)} \to +\infty$. □

4. Odd heteroclinic solutions

We focalize now in the study of heteroclinic solutions which are odd in the first variable, hence we will consider a subset of $\Gamma(-v, v)$, $v \in \mathcal{M}_0$, so let us introduce the set

$$E^{odd} = \{u \in E \mid u \text{ is odd with respect to } x_1\},$$

In what follows, when we will consider functions $u \in E^{odd}$ we often present their properties for $x_1 \geq 0$, avoiding to write the corresponding ones for $x_1 < 0$. In particular, for every $u \in E^{odd}$ we have $J(u) = 2J^+(u)$, where

$$J^+(u) = \sum_{p \geq 0} J_{p,0}(u)\,.$$

For any $v \in \mathcal{M}_0$ let

$$\Gamma(v) = \{u \in E^{odd} \mid \|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \to 0 \text{ as } p \to +\infty\} \subseteq \Gamma(-v, v)\,.$$

In this setting we can rewrite Lemma 3.7 as follows.

Lemma 4.1. *For every $u \in E^{odd}$ for which $J(u) < +\infty$ there exists $v \in \mathcal{M}_0$ such that $\|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \to 0$ as $p \to +\infty$, that is $u \in \Gamma(v)$.*

We are going to look for minimizer of J in the set $\Gamma(v)$. So, for every $v\in\mathcal{M}_0$ we set

$$c(v)=\inf_{u\in\Gamma(v)}J(u)\quad\text{and}\quad\mathcal{M}(v)=\{u\in\Gamma(v)\mid J(u)=c(v)\}\,. \tag{4.1}$$

Notice that for any $v\in\mathcal{M}_0$ we have $c(-v,v)\le c(v)<+\infty$ holds and, by Lemma 3.8 since by (N_0), $0\notin\mathcal{M}_0$, we have the following.

Lemma 4.2. *For any $v\in\mathcal{M}_0$, $c(v)\ge\frac{\lambda_0}{2}$, and $c(v)\to+\infty$ as $\|v\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)}\to+\infty$.*

Moreover, note that, by assumption (N_0), the intersection between $\mathcal{M}_0$ and a bounded set consists of a finite number of elements. Hence, from the previous lemma, the minimum

$$c=\min_{v\in\mathcal{M}_0}c(v) \tag{4.2}$$

is well defined and the set

$$\mathcal{M}_0^{min}=\{v\in\mathcal{M}_0\mid c(v)=c\} \tag{4.3}$$

is nonempty and consists of a finite number of elements. In particular, we have

$$\min_{v\in\mathcal{M}_0\setminus\mathcal{M}_0^{min}}c(v)>c\,. \tag{4.4}$$

The following lemma provides a concentration property for $u\in E^{odd}$ such that $J(u)$ is *close to* the value c: the elements of the sequence $(u(\cdot+p))_{p\in\mathbb{Z}}$ remain *far from* $\mathcal{M}_0$ only for a finite number of indexes p. Moreover, $(u(\cdot+p))_{p\in\mathbb{Z}}$ approaches an element $v_0\in\mathcal{M}_0$ only once. Indeed, recalling the notation introduced in Remark 3.2, we have

Lemma 4.3. *For any $r\in(0,r_1]$ there exists $\ell(r)\in\mathbb{N}$, $\delta(r)\in(0,\frac{r_0}{4})$ with $\delta(r)\to0$ as $r\to0^+$ with the following property: if $u\in E^{odd}$ is such that $J(u)\le c+\Lambda(r)$ then*

- *(i) if $\mathrm{dist}_{W^{1,2}(T_{p,0},\mathbb{R}^m)}(u,\mathcal{M}_0)\ge r$ for every p in a set $\mathcal{I}$ of consecutive integers, then $\mathrm{Card}(\mathcal{I})\le\ell(r)$,*
- *(ii) if $\|u-v_0\|_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)}\le r$ for some index $p_0\ge0$ and $v_0\in\mathcal{M}_0$, then $\|u-v_0\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)}\le\delta(r)$ for every $p\ge p_0$, and $\sum_{p=p_0}^{+\infty}J_{p,0}(u)\le2\Lambda(r)$.*

Proof. Note that (i) plainly follows from Lemma 2.1-(3), setting $\ell(r)=\left[\frac{c+\Lambda(r)}{\beta(r)}\right]+1$, where $[\cdot]$ denotes the integer part.
To prove (ii), we consider $\tilde u\in E^{odd}$ defined for $x_1\ge0$ as

$$\tilde u(x_1,y)=\begin{cases}u(x_1,y) & \text{if } x_1\in[0,p_0],\\ u(x_1,y)(p_0+1-x_1)+v_0(x_1,y)(x_1-p_0) & \text{if } x_1\in(p_0,p_0+1),\\ v_0(x_1,y) & \text{if } x_1\in[p_0+1,+\infty)\end{cases}$$

Hence, $\tilde u\in\Gamma(v_0)$ and since $\tilde u\equiv u$ in $[-p_0,p_0]\times\mathbb{R}^{n-1}$, while $\tilde u=v_0$ in $[p_0+1,+\infty)\times\mathbb{R}^{n-1}$, we obtain

$$\tfrac12c\le\tfrac12c(v_0)\le\tfrac12J(\tilde u)=J^+(\tilde u)=J^+(u)-\sum_{p=p_0}^{+\infty}J_{p,0}(u)+J_{p_0,0}(\tilde u).$$

By definition, on $T_{p_0,0}$ we have $\tilde u(x_1,y)-v_0(x_1,y)=(p_0+1-x_1)(u(x_1,y)-v_0(x_1,y))$ and so $\|\tilde u-v_0\|_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)}\le2\|u-v_0\|_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)}\le2r$. Since $\tilde u=v_0$ in $[p_0+1,p_0+2]\times\mathbb{R}^{n-1}$, we have $\mathrm{dist}_p(\tilde{u}, \mathcal{M}_0) = \|\tilde{u} - v_0\|_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)} \leq 2r$, so that, by Remark 3.2, we obtain $J_{p_0,0}(\tilde{u}) \leq \Lambda(r) \leq \frac{\lambda_0}{8}$ and therefore

$$\tfrac{1}{2}c \leq \tfrac{1}{2}J(\tilde{u}) \leq J^+(u) - \sum_{p=p_0}^{+\infty} J_{p,0}(u) + \Lambda(r) \leq \tfrac{1}{2}c - \sum_{p=p_0}^{+\infty} J_{p,0}(u) + \tfrac{3}{2}\Lambda(r).$$

Then $\sum_{p=p_0}^{+\infty} J_{p,0}(u) \leq \frac{3}{2}\Lambda(r)$ and in particular $J_{p,0}(u) \leq \frac{3}{2}\Lambda(r) \leq \frac{\lambda_0}{2}$ for any $p \geq p_0$. Hence, by Lemma 3.3, $\|u - v_0\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq \rho(3\Lambda(r)) < r_0$ for any $p \geq p_0$. Hence (ii) follows setting $\delta(r) = \rho(3\Lambda(r))$. Indeed, by Remark 3.2, we have $\lim_{r\to 0^+} \delta(r) = 0$ and, since $\Lambda(r) \leq \frac{\lambda_0}{8}$ for all $r \in (0, r_1]$, we get $\delta(r) \leq \rho(\lambda_0) \leq \frac{r_0}{4}$ for every $r \in (0, r_1)$. □

By the previous lemma we get

Lemma 4.4. *For any $r \in (0, r_1]$, if $u \in E^{odd}$ satisfies $J(u) \leq c + \Lambda(r)$, then there exists $v_0 \in \mathcal{M}_0$ such that $u \in \Gamma(v_0)$ and*

- (i) *if $\|u - v_0\|_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)} \leq r$ for a certain index $p_0 \geq 0$, then $\|u - v_0\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq \delta(r)$ for every $p \geq p_0$, and $\sum_{p=p_0}^{+\infty} J_{p,0}(u) \leq 2\Lambda(r)$.*
- (ii) *if $w \in \mathcal{M}_0 \setminus \{v_0\}$, then $\|u - w\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} > r_1$ for every $p \in \mathbb{Z}$, $p \geq 0$.*

Proof. Note that the existence of v_0 such that $u \in \Gamma(v_0)$ is ensured by Lemma 4.1 and (i) plainly follows from Lemma 4.3-(ii). To prove (ii) we argue by contradiction assuming that there exist $\bar{p}_0 \in \mathbb{Z}$, $\bar{p}_0 \geq 0$ and $w \in \mathcal{M}_0 \setminus \{v_0\}$ such that $\|u - w\|_{W^{1,2}(T_{\bar{p}_0,0},\mathbb{R}^m)} \leq r_1$. Again, by Lemma 4.3-$(ii)$ we get $\|u - w\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq \delta(r_1) \leq \frac{r_0}{4}$ for every $p \geq \bar{p}_0$ which is in contradiction with $u \in \Gamma(v_0)$, recalling the definition of r_0 in (2.4). □

As a direct consequence of Lemmas 4.3 and 4.4 we obtain the following concentration result.

Lemma 4.5. *For any $\rho \in (0, r_1]$ there exists $\tilde{\Lambda}(\rho)$, with $\tilde{\Lambda}(\rho) \to 0$ as $\rho \to 0^+$, and $\tilde{\ell}(\rho) \in \mathbb{N}$ such that if $u \in E^{odd}$ satisfies $J(u) \leq c + \tilde{\Lambda}(\rho)$, then there exists $v_0 \in \mathcal{M}_0$ such that $u \in \Gamma(v_0)$ and*

- (i) *$\|u - v_0\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq \rho$ for every $p \geq \tilde{\ell}(\rho)$;*
- (ii) *$\sum_{p=\tilde{\ell}(\rho)}^{+\infty} J_{p,0}(u) \leq 2\tilde{\Lambda}(\rho)$;*
- (iii) *$\|u - w\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \geq r_1$ for every $p \in \mathbb{Z}$, $p \geq 0$ and $w \in \mathcal{M}_0 \setminus \{v_0\}$.*

Proof. The existence of v_0 such that $u \in \Gamma(v_0)$ is again ensured by Lemma 4.1. By Lemma 4.3, given any $\rho \in (0, r_1]$, there exists $r \in (0, \rho)$ such that $\delta(r) \leq \rho$. Then, if $u \in \Gamma(v_0)$ is such that $J(u) \leq c + \Lambda(r)$, by Lemma 4.3-(i), there exists $p_0 \in [0, \ell(r) + 1]$ such that $\mathrm{dist}_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)}(u, \mathcal{M}_0) < r$ and hence a $v \in \mathcal{M}_0$ such that $\|u - v\|_{W^{1,2}(T_{p_0,0},\mathbb{R}^m)} < r$. Therefore, by Lemma 4.3-(ii), we obtain $\|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} < \delta(r)$ for all $p \geq p_0$ and since $\delta(r) < \rho < r_1 < \frac{r_0}{4}$, we can conclude that $v \equiv v_0$ and hence that $\|u - v_0\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq \rho$ for every $p \geq p_0$. Moreover, again by Lemma 4.3-(ii), we have $\sum_{p=p_0}^{+\infty} J_{p,0}(u) \leq 2\Lambda(r)$. Hence (i) and (ii) follows setting $\tilde{\ell}(\rho) = \ell(r) + 1$ and $\tilde{\Lambda}(\rho) = \Lambda(r)$.
Finally, $\|u - w\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \geq r_1$ for every $p \in \mathbb{Z}$, $p \geq 0$, and $w \in \mathcal{M}_0 \setminus \{v_0\}$ follows directly by Lemma 4.4 -(ii). □

We are now able to prove the existence of a minimum of J in the set $\Gamma(v)$ for every $v \in \mathcal{M}_0^{min}$, i.e., that $\mathcal{M}(v) \neq \varnothing$ for all $v \in \mathcal{M}_0^{min}$.

Theorem 4.6. *Let $v \in \mathcal{M}_0^{min}$, then there exists $u \in \Gamma(v)$ such that $J(u) = c(v) = c$.*

Proof. Let $(u_k)_k \subset \Gamma(v)$ be such that $J(u_k) \to c(v)$. Without loss of generality we can assume that $J(u_k) \leq c + \tilde{\Lambda}(r_1)$ for any $k \in \mathbb{N}$. By Lemma 4.5, we obtain that for any $k \in \mathbb{N}$,

$$\|u_k - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq r_1 \text{ for every } p \geq \tilde{\ell}(r_1). \tag{4.5}$$

By Lemma 3.6, since E^{odd} is weakly closed, there exists $u \in E^{odd}$ such that, along a subsequence, $u_k \to u$ weakly in E^{odd}. Finally, by (4.5) and the weakly lower semicontinuity of the distance we obtain

$$\|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \leq r_1 \text{ for every } p \geq \tilde{\ell}(r_1). \tag{4.6}$$

Therefore, by Lemma 3.7, we conclude that $\|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \to 0$ as $p \to +\infty$, so that $u \in \Gamma(v)$. Finally, by semicontinuity, $J(u) = c(v)$. □

By Theorem 4.6 we know that for every $v_0 \in \mathcal{M}_0^{min}$, $\mathcal{M}(v_0)$ is nonempty. One can prove that $\mathcal{M}(v_0)$ consists of weak solutions of (PDE).

Lemma 4.7. *Given $\bar{u} \in \mathcal{M}(v_0)$, with $v_0 \in \mathcal{M}_0^{min}$, then for any $\psi \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m)$ we have*

$$\int_{\mathbb{R}\times[0,1]^{n-1}} \nabla\bar{u} \cdot \nabla\psi + F_u(x, \bar{u})\psi \, dx = 0 \, .$$

The proof can be adapted by the one of Lemma 3.3 of [4] or Lemma 5.2 of [6]. Therefore we get that any $u \in \mathcal{M}(v_0)$ is a classical $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$ solution of (PDE) which is 1-periodic in the variables x_i, $i \geq 2$.

Finally, we now study further compactness properties for the functional J that will be useful in the next section. They will be obtained as consequences of the nondegeneracy property of the elements of $\mathcal{M}_0$ asked in (N). In particular assumption (N) asks that, for every $v \in \mathcal{M}_0$, the linearized operator about v

$$L_v : W^{2,2}([0,1]^n, \mathbb{R}^m) \subset L^2([0,1]^n, \mathbb{R}^m) \to L^2([0,1]^n, \mathbb{R}^m) \, ,$$

$$L_v h = -\Delta h + F_{u,u}(\cdot, v(\cdot))h$$

has spectrum which does not contain 0. This is the assumption made in [30] and it is indeed equivalent to require as in (N) that

(N_1) there exists $\alpha_0 > 0$ such that

$$J_0''(v)h \cdot h = \int_{[0,1]^n} |\nabla h(x)|^2 + F_{u,u}(x, v(x))|h(x)|^2 \, dx \geq \alpha_0 \|h\|^2_{L^2([0,1]^n,\mathbb{R}^m)}$$

for every $h \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ and every $v \in \mathcal{M}_0$.

As a consequence of (N_1) we obtain the following (see also Lemma 3.6 in [2]).

Lemma 4.8. *There exist $r_2 \in (0, r_1)$ and $\omega_1 > \omega_0 > 0$ such that if $u \in W^{1,2}(T_{p,0}, \mathbb{R}^m)$, $p \in \mathbb{Z}$, verifies $\|u - v\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \le r_2$ for some $v \in \mathcal{M}_0$ then*

$$\omega_0 \|u - v\|^2_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \le J_{p,0}(u) \le \omega_1 \|u - v\|^2_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \,. \tag{4.7}$$

Proof. Notice that, by (N_1), if $h \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ and $v \in \mathcal{M}_0$ then

$$\begin{aligned}\int_{[0,1]^n} |\nabla h(x)|^2 + F_{u,u}(x, v(x))|h(x)|^2\, dx &\ge \alpha_0 \|h\|^2_{L^2([0,1]^n,\mathbb{R}^m)} \\ &\ge -\alpha_0 f_0 \int_{[0,1]^n} F_{u,u}(x, v(x))|h(x)|^2\, dx,\end{aligned}$$

where $f_0 = 1/\|F_{uu}\|_\infty$, and so

$$\int_{[0,1]^n} \frac{1}{1+\alpha_0 f_0} |\nabla h(x)|^2\, dx + \int_{[0,1]^n} F_{u,u}(x, v(x))|h(x)|^2\, dx \ge 0\,.$$

We conclude that

$$J_0''(v)h \cdot h = \int_{[0,1]^n} |\nabla h(x)|^2 + F_{u,u}(x, v(x))|h(x)|^2\, dx \ge \frac{\alpha_0 f_0}{1+\alpha_0 f_0} \|\nabla h\|^2_{L^2([0,1]^n,\mathbb{R}^m)}$$

and so, using (N_1) and setting $\omega_0 = \frac{\alpha_0}{6} \min\{1, \frac{f_0}{1+\alpha_0 f_0}\}$, we obtain

$$J_0''(v)h \cdot h \ge 3\omega_0 \|h\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)}, \quad \forall\, h \in W^{1,2}([0,1]^n, \mathbb{R}^m).$$

Since by Taylor's formula we have $J_0(u) - c_0 = \frac{1}{2} J_0''(v)(u-v)\cdot(u-v) + o(\|u-v\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)})$ for all $v \in \mathcal{M}_0$ and $u \in W^{1,2}([0,1]^n, \mathbb{R}^m)$, we obtain that there exists $r_2 \in (0, \frac{r_1}{4})$ such that if $u \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ verifies $\|u - v\|_{W^{1,2}([0,1]^n,\mathbb{R}^m)} \le r_2$ for some $v \in \mathcal{M}_0$, then

$$J_0(u) - c_0 \ge \omega_0 \|u - v\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)}. \tag{4.8}$$

On the other hand, again Taylor's expansion gives us

$$\begin{aligned} J_0(u) - c_0 &= \tfrac{1}{2} J_0''(v)(u-v)\cdot(u-v) + o(\|u-v\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)}) \\ &= \tfrac{1}{2}\|\nabla(u-v)\|^2_{L^2([0,1]^n,\mathbb{R}^m)} + \tfrac{1}{2}\int_{[0,1]^n} F_{u,u}(x, v(x))|u(x)-v(x)|^2\, dx \\ &\quad + o(\|u-v\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)}) \\ &\le \tfrac{1}{2}\|\nabla(u-v)\|^2_{L^2([0,1]^n,\mathbb{R}^m)} + \tfrac{1}{2f_0}\|u-v\|^2_{L^2([0,1]^n,\mathbb{R}^m)} + o(\|u-v\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)}) \end{aligned}$$

and we deduce that there exists $\omega_1 > \omega_0$ such that, taking r_2 smaller if necessary, if $u \in W^{1,2}([0,1]^n, \mathbb{R}^m)$ verifies $\|u - v\|_{W^{1,2}([0,1]^n)} \le r_2$, $v \in \mathcal{M}_0$, then

$$J_0(u) - c_0 \le \omega_1 \|u - v\|^2_{W^{1,2}([0,1]^n,\mathbb{R}^m)}. \tag{4.9}$$

The lemma follows by periodicity from (4.8) and (4.9) recalling that $T_{p,0} = [p, p+1] \times [0,1]^{n-1}$ and that $J_{p,0}(u) = J_0(u(\cdot + p)) - c_0$ for all $u \in W^{1,2}(T_{p,0}, \mathbb{R}^m)$. $\square$

We emphasize that, for our purposes, we need (N_1) only for the choice $v \in \mathcal{M}_0^{min}$.

Remark 4.9. In connection with Remark 2.5, arguing as in Remark 3.8 of [2], we can prove that (4.7) holds true also for the functional $J_{\sigma_0}(u) = \int_{\sigma_0} L(u)\,dx - c_0$ on $W^{1,2}(\sigma_0, \mathbb{R}^m)$, that is, if $\|u - v\|_{W^{1,2}(\sigma_0,\mathbb{R}^m)} \le r_1$ for some $v \in \mathcal{M}_0$ then

$$\omega_0 \|u - v\|^2_{W^{1,2}(\sigma_0,\mathbb{R}^m)} \le J_{\sigma_0}(u) \le \omega_1 \|u - v\|^2_{W^{1,2}(\sigma_0,\mathbb{R}^m)}. \tag{4.10}$$

Hence, recalling the definition (4.1), plainly adapting the proof of Lemma 3.10 in [2], we obtain

Lemma 4.10. *Let $v_0 \in \mathcal{M}_0^{min}$ and $(u_k)_k \subset \Gamma(v_0)$ be such that $J(u_k) \to c$. Then there exists $u \in \mathcal{M}(v_0)$ such that, up to a subsequence, $\|u_k - u\|_{W^{1,2}(\mathbb{R}\times[0,1]^{n-1},\mathbb{R}^m)} \to 0$ as $k \to +\infty$.*

5. Saddle type solutions

In this section we prove our main theorem. To this aim, following and adapting the argument in [2], we will first prove the existence of a solution of (PDE) on the unbounded *triangle*

$$\mathcal{T} = \{x \in \mathbb{R}^n \mid x_2 \ge |x_1|\}$$

satisfying Neumann boundary conditions on $\partial\mathcal{T}$, which is odd in the first variable x_1, asymptotic as $x_2 \to +\infty$ to a certain heterocline $v_0 \in \mathcal{M}$ where

$$\mathcal{M} := \bigcup_{v \in \mathcal{M}_0^{min}} \mathcal{M}(v).$$

Then, by recursive reflections with respect to the hyperplanes $x_2 = \pm x_1$, we will recover a solution of (PDE) on the whole $\mathbb{R}^n$.

Let us introduce now some notations. We define the *squares*

$$T_{p,k} := [p, p+1] \times [k, k+1] \times [0,1]^{n-2}, \quad p \in \mathbb{Z},\ k \in \mathbb{N}$$

and the *horizontal strips*

$$\mathcal{S}_k := \mathbb{R} \times [k, k+1] \times [0,1]^{n-2} = \bigcup_{p\in\mathbb{Z}} T_{p,k}, \quad k \in \mathbb{N}$$

The intersection between the strip $\mathcal{S}_k$ and the triangle $\mathcal{T}$ consists of a *bounded strip*

$$\mathcal{T}_k := \mathcal{S}_k \cap \mathcal{T} = \left(\bigcup_{p=-k}^{k-1} T_{p,k} \right) \cup \tau_k$$

where $\tau_k = \{x \in T_{k,k} \cup T_{-k-1,k} \mid x_2 \ge |x_1|\}$.

For every $k \in \mathbb{N}$ we define the sets of functions

$$E_k = \{u \in W^{1,2}(\mathcal{T}_k, \mathbb{R}^m) \mid u \text{ is odd in } x_1, \text{ 1-periodic in } x_3, ..., x_n\}$$

and the normalized functionals on the bounded strips $\mathcal{T}_k$ as

$$J_k(u) = \int_{\mathcal{T}_k} L(u)\,dx - (2k+1)c_0 = \sum_{p=-k}^{k-1} J_{p,k}(u) + \int_{\tau_k} L(u)\,dx - c_0, \quad k \in \mathbb{N},$$

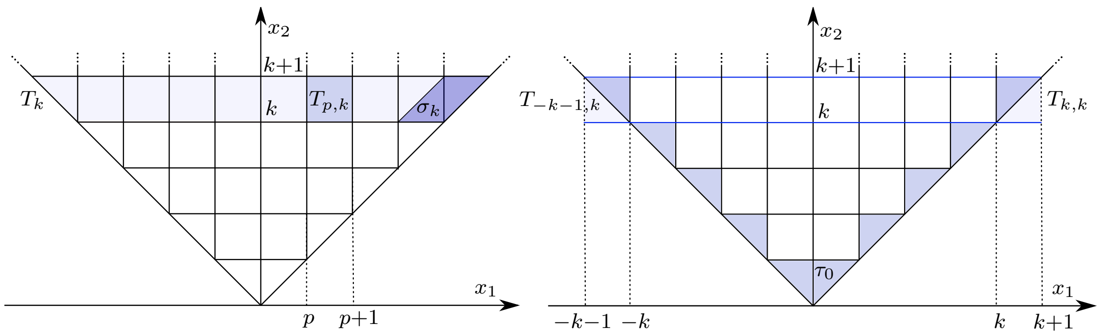

FIGURE 1. *The decomposition of the triangular set $\mathcal{T}$*

for every $u \in E_k$, where $J_{p,k}(u) = \int_{T_{p,k}} L(u)\, dx - c_0$.

Remark 5.1. Notice that $J_k(u) \geq 0$ for every $u \in E_k$, $k \in \mathbb{N}$. Indeed, we can view the restriction $u|_{T_{p,k}}$ as a traslation of a function in $W^{1,2}([0,1]^n, \mathbb{R}^m)$ and the restriction on $u|_{\tau_k}$ can be treated similarly using Lemma 2.4, the symmetry of u and Remark 2.5. Moreover, we note that the functional J_k is lower semicontinuous with respect to the weak $W^{1,2}(\mathcal{T}_k, \mathbb{R}^m)$ topology for every $k \in \mathbb{N}$.

Then, we can set

$$c_k = \inf_{E_k} J_k(u) \quad \text{and} \quad \mathcal{M}_k = \{u \in E_k \mid J_k(u) = c_k\}.$$

We plainly obtain that $\mathcal{M}_k \neq \varnothing$ and that the sequence $(c_k)_k$ is increasing. Moreover, $c_k \leq c$, evaluating J_k on a function $u \in \mathcal{M}(v_0)$ with $v_0 \in \mathcal{M}_0^{min}$. Moreover, the non degeneracy assumption (N_1) permits us to obtain the following stronger result.

Lemma 5.2. *We have $\sum_{k=0}^{\infty}(c - c_k) < +\infty$, in particular $c_k \to c$ as $k \to +\infty$.*

Proof. Denoted by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ the canonical basis of $\mathbb{R}^n$ and fixed $k \in \mathbb{N}$, given $u_k \in \mathcal{M}_k$ we consider the traslation $u_k^{\downarrow} = u_k(x + \boldsymbol{e}_2) \in E_{k-1}$. We obtain

$$\begin{aligned}
c_{k-1} \leq J_{k-1}(u_k^{\downarrow}) &= 2\sum_{p=0}^{k-2} J_{p,k-1}(u_k^{\downarrow}) + \int_{\tau_{k-1}} L(u_k^{\downarrow})\, dx - c_0 \\
&= 2\sum_{p=0}^{k-2} J_{p,k}(u_k) + \int_{\tau_{k-1}+\boldsymbol{e}_2} L(u_k)\, dx - c_0 \\
&= J_k(u_k) - 2\left(\int_{\sigma_k} L(u_k)\, dx - c_0\right) = c_k - 2\left(\int_{\sigma_k} L(u_k)\, dx - c_0\right).
\end{aligned}$$

where, for every $k \in \mathbb{N}$,

$$\sigma_k = \{x \in \mathcal{T}_k \mid x_2 - 1 \leq x_1 \leq x_2\} = \sigma_0 + k(\boldsymbol{e}_1 + \boldsymbol{e}_2).$$

Setting $J_{\sigma_k}(u) = \int_{\sigma_k} L(u)\, dx - c_0$, the above inequality gives

$$c_k \geq c_{k-1} + 2J_{\sigma_k}(u_k)$$

so that

$$c \geq c_k \geq c_{k-1} + 2J_{\sigma_k}(u_k) \geq c_1 + 2\sum_{\iota=2}^{k} J_{\sigma_\iota}(u_\iota)\,.$$

Hence, since by periodicity and Remark 2.5 we have $J_{\sigma_\iota}(u) \geq 0$ for any $u \in W^{1,2}(\sigma_\iota, \mathbb{R}^m)$ and $\iota \in \mathbb{N}$, we conclude that $\sum_{\iota=2}^{\infty} J_{\sigma_\iota}(u_\iota) < +\infty$. In particular $\lim_{k\to+\infty} J_{\sigma_k}(u_k) = 0$. By Remark 2.5 we can find k_0 large enough to guarantee that for every $k \geq k_0$ there exists $v_k \in \mathcal{M}_0$ such that $\|u_k - v_k\|_{W^{1,2}(\sigma_k,\mathbb{R}^m)} \leq \frac{r_2}{4}$. Let $\tilde{u}_k : \mathcal{S}_k \to \mathbb{R}^m$ (for $k > k_0$) be defined as

$$\tilde{u}_k(x) = \begin{cases} u_k(x_1, x_2, y) & \text{if } 0 \leq x_1 \leq x_2 - 1\,, \\ u_k(x_1, x_2, y)(x_2 - x_1) + v_k(x_1, x_2, y)(x_1 - x_2 + 1) & \text{if } x_2 - 1 \leq x_1 \leq x_2\,, \\ v_k(x_1, x_2, y) & \text{if } x_2 \leq x_1 \\ \text{odd extended for } x_1 < 0 \end{cases}$$

where $y = (x_3, \ldots, x_n)$, and note that $\|\tilde{u}_k - v_k\|_{W^{1,2}(\sigma_k,\mathbb{R}^m)} \leq 2\|u_k - v_k\|_{W^{1,2}(\sigma_k,\mathbb{R}^m)}$. Hence $\|\tilde{u}_k - v_k\|_{W^{1,2}(\sigma_k,\mathbb{R}^m)} \leq r_2$ and we can use (4.10) and the periodicity to find

$$J_{\sigma_k}(\tilde{u}_k) \leq \omega_1 \|\tilde{u}_k - v_k\|^2_{W^{1,2}(\sigma_k,\mathbb{R}^m)} \leq 4\omega_1 \|u_k - v_k\|^2_{W^{1,2}(\sigma_k,\mathbb{R}^m)} \leq 4\tfrac{\omega_1}{\omega_0} J_{\sigma_k}(u_k)\,.$$

Then

$$\sum_{\ell=2}^{+\infty} J_{\sigma_\ell}(\tilde{u}_\ell) \leq 4\tfrac{\omega_1}{\omega_0} \sum_{\ell=2}^{+\infty} J_{\sigma_\ell}(u_\ell) < +\infty\,. \tag{5.1}$$

Setting $\tilde{u}_k^{\downarrow}(x) := \tilde{u}_k(x + k\boldsymbol{e}_2)$, we have $\tilde{u}_k^{\downarrow} \in \Gamma(v_k)$, so that we finally obtain

$$\begin{aligned} c \leq c(v_k) \leq J(\tilde{u}_k^{\downarrow}) &= J_k(\tilde{u}_k) = J_k(u_k) + 2J_{\sigma_k}(\tilde{u}_k) - 2J_{\sigma_k}(u_k) \\ &= c_k + 2J_{\sigma_k}(\tilde{u}_k) - 2J_{\sigma_k}(u_k) \leq c_k + 2J_{\sigma_k}(\tilde{u}_k)\,. \end{aligned}$$

Then, $0 \leq c - c_k \leq 2J_{\sigma_k}(\tilde{u}_k)$ and lemma follows by (5.1). □

We can now introduce on the set

$$\mathcal{E} = \{u \in W^{1,2}_{loc}(\mathcal{T}, \mathbb{R}^m) \mid u \text{ is odd in } x_1,\ u_i(x) \geq 0 \text{ for } x_1 \geq 0\,, \forall i = 1, ..., m\}.$$

the functional

$$\mathcal{J}(u) = \sum_{k=0}^{+\infty} \left(J_k(u) - c_k\right)\,.$$

Notice that $\mathcal{J}(u) \geq 0$ for every $u \in \mathcal{E}$. Indeed, the restriction $u|_{\mathcal{T}_k} \in E_k$ and so $J_k(u) \geq c_k$ for any $k \in \mathbb{N}$. Moreover, $\mathcal{J}$ is lower semicontinuous in the weak topology of $W^{1,2}_{loc}(\mathcal{T}, \mathbb{R}^m)$. Lemma 5.2 is crucial in order to show that $\mathcal{J}$ is finite for at least one $u \in \mathcal{E}$.

Lemma 5.3. *If $u \in \mathcal{M}(v_0)$ for some $v_0 \in \mathcal{M}_0^{min}$, then $\mathcal{J}(u) < +\infty$.*

Proof. Let $u \in \mathcal{M}(v_0)$ for some $v_0 \in \mathcal{M}_0^{min}$, then its restriction on $\mathcal{T}$ belongs to $\mathcal{E}$ and $J(u) = c(v_0) = c$. We have $J_k(u) - c_k \leq J(u) - c_k = c - c_k$. By Lemma 5.2, we have $\mathcal{J}(u) = \sum_{k=0}^{+\infty} (J_k(u) - c_k) \leq \sum_{k=0}^{+\infty} (c - c_k) < +\infty$. □

We now look for a minimum of the functional $\mathcal{J}$ on $\mathcal{E}$, thus we set

$$\tilde{c} = \inf_{\mathcal{E}} \mathcal{J}(u) \quad \text{and} \quad \widetilde{\mathcal{M}} = \{u \in \mathcal{E} \mid \mathcal{J}(u) = \tilde{c}\}\,.$$

Lemma 5.3, gives that $\tilde{c} \in \mathbb{R}$ and we can prove the existence of the minimum applying standard arguments.

Proposition 5.4. *We have $\widetilde{\mathcal{M}} \neq \varnothing$.*

Proof. Let $(u_j)_j \subset \mathcal{E}$ be a minimizing sequence for $\mathcal{J}$. Fix $r \in \mathbb{N}$ and consider $S_r = \mathcal{T} \cap \{x_2 \le r\}$. We can obtain a bound to the norm of the derivative

$$\begin{aligned}
\|\nabla u_j\|^2_{L^2(S_r,\mathbb{R}^m)} &= \sum_{k=0}^{r-1} \|\nabla u_j\|^2_{L^2(\mathcal{T}_k,\mathbb{R}^m)} \le 2\sum_{k=0}^{r-1}[J_k(u_j) + (2k+1)c_0] \\
&= 2\sum_{k=0}^{r-1}[J_k(u_j) - c_k] + 2\sum_{k=0}^{r-1}[(2k+1)c_0 + c_k] \\
&\le 2\mathcal{J}(u_j) + 2r[(2r-1)c_0 + c]\,.
\end{aligned}$$

from which, since $u_j \equiv 0$ on $\{x_1 = 0\} \cap \mathcal{T}$, by Fubini Theorem and Hölder inequality, we get that $(u_j)_j$ is bounded in $W^{1,2}(S_r, \mathbb{R}^m)$. Therefore, up to a subsequence, it converges weakly in $W^{1,2}(S_r, \mathbb{R}^m)$. Hence, by a diagonal argument, we can find $w \in W^{1,2}_{loc}(\mathcal{T}, \mathbb{R}^m)$ such that, up to a subsequence, $u_j \to w$ weakly in $W^{1,2}_{loc}(\mathcal{T}, \mathbb{R}^m)$ and almost everywhere in $\mathcal{T}$. Hence, $w \in \mathcal{E}$ and, by semicontinuity, $\mathcal{J}(w) = \tilde{c}$ thus finishing the proof. □

Arguing as in [2, 4, 6] (see e.g. the argument in Lemma 3.3 of [4] or Lemma 5.2 of [6]), we can prove that if $u \in \widetilde{\mathcal{M}}$ then it is a weak solution of (PDE) on $\mathcal{T}$ with Neumann boundary condition on $\partial\mathcal{T}$. Then we can conclude that every $u \in \widetilde{\mathcal{M}}$ is indeed a classical $\mathcal{C}^2$ solution of (PDE). Finally, using (F_3), we can recursively reflect w with respect to the hyperplanes $x_2 = \pm x_1$, obtaining an entire solution w of (PDE) (see e.g. [2]). By construction, it is odd both in x_1 and x_2, symmetric with respect to the hyperplanes $x_1 = \pm x_2$ and it is 1–periodic in $x_3, ..., x_n$. Hence, it satisfies hypotheses (ii)-(iii) of Theorem 1.1.

In the next lemma we prove the asymptotic behavior of the solution w.

Lemma 5.5. *Let $w \in W^{1,2}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ be the function obtained by recursive reflection of a given $w_0 \in \widetilde{\mathcal{M}}$. Then there exists $\bar{v} \in \mathcal{M}_0^{min}$ such that*

$$\lim_{k\to+\infty} \text{dist}_{W^{1,2}(\mathcal{T}_k,\mathbb{R}^m)}(w, \mathcal{M}(\bar{v})) = 0.$$

Proof. Let w be as in the statement, we start proving that there exists $\bar{v} \in \mathcal{M}_0^{min}$ such that

$$\lim_{k\to+\infty} \|w - \bar{v}\|_{W^{1,2}(T_{k,k},\mathbb{R}^m)} = 0\,. \tag{5.2}$$

We have $\mathcal{J}(w) = \mathcal{J}(w_0) = \tilde{c} < +\infty$. Hence, $J_k(w) - c_k \to 0$ as $k \to +\infty$ so that, by Lemma 5.2, $J_k(w) \to c$ as $k \to +\infty$. Therefore, we can find a sequence $(p_k)_{k\in\mathbb{N}}$, with $p_k \in [0, k-1] \cap \mathbb{N}$ such that $J_{p_k,k}(w) \to 0$ as $k \to +\infty$, and in particular $J_0(w(\cdot + p_k \boldsymbol{e}_1 +$

$k\,\boldsymbol{e}_2)) \to c_0$. By Lemma 2.1-(3), we get $\mathrm{dist}_{W^{1,2}(T_{p_k,k},\mathbb{R}^m)}(w,\mathcal{M}_0) = \mathrm{dist}_{W^{1,2}([0,1]^n,\mathbb{R}^m)}(w(\cdot + p_k\,\boldsymbol{e}_1 + k\,\boldsymbol{e}_2),\mathcal{M}_0) \to 0$ as $k\to+\infty$ thus giving the existence of $v_k\in\mathcal{M}_0$ such that

$$\|w - v_k\|_{W^{1,2}(T_{p_k,k},\mathbb{R}^m)} \to 0\,,\ \text{as } k\to+\infty.$$

Now, for every $k\in\mathbb{N}$, we define in the horizontal strip $\mathcal{S}_k$ the following interpolation between w and v_k:

$$w_k(x_1,x_2,y) = \begin{cases} w(x_1,x_2,y) & \text{if } 0\le x_1\le p_k \\ w(x_1,x_2,y)(p_k - x_1 + 1) + v_k(x_1,x_2,y)(x_1 - p_k) & \text{if } p_k < x_1 \le p_k+1 \\ v_k(x_1,x_2,y) & \text{if } x_1 > p_k+1 \\ \text{odd extended for } x_1<0 \end{cases}$$

A computation gives $\|w_k - v_k\|_{W^{1,2}(T_{p_k,k},\mathbb{R}^m)} \le 2\|w - v_k\|_{W^{1,2}(T_{p_k,k},\mathbb{R}^m)} \to 0$ so that

$$\lim_{k\to+\infty} J_{p_k,k}(w_k) = 0\,.$$

Now, consider $w_k^{\downarrow}(x) = w_k(x + k\,\boldsymbol{e}_2)$ defined on $\mathcal{S}_0$. We have $w_k^{\downarrow}\in\Gamma(v_k)$, therefore

$$c \le J(w_k^{\downarrow}) = 2\sum_{p=0}^{p_k} J_{p,0}(w_k^{\downarrow}) = 2\sum_{p=0}^{p_k-1} J_{p,k}(w) + 2J_{p_k,k}(w_k) \le J_k(w) + 2J_{p_k,k}(w_k)\,.$$

and hence, since $J_k(w)\to c$ and $J_{p_k,k}(w_k)\to 0$, we obtain $J(w_k^{\downarrow})\to c$ as $k\to+\infty$. As a consequence, since $w_k^{\downarrow}\in\Gamma(v_k)$, by (4.4), we can conclude that $v_k\in\mathcal{M}_0^{min}$. Moreover we have

$$J_k(w) - J_k(w_k) = 2\sum_{p=p_k}^{k-1} J_{p,k}(w) + \int_{\tau_k} L(w)\,dx - c_0 - 2J_{p_k,k}(w_k)$$

and since $J_k(w)\to c$, $J_k(w_k) = J(w_k^{\downarrow})\to c$ and $J_{p_k,k}(w_k)\to 0$, we obtain

$$2\sum_{p=p_k}^{k-1} J_{p,k}(w) + \int_{\tau_k} L(w)\,dx - c_0 \to 0\,,\ \text{as } k\to+\infty\,. \tag{5.3}$$

In particular $\int_{\tau_k} L(w)\,dx - c_0 \to 0$, so that $\lim_{k\to+\infty} J_{k,k}(w) = 0$, by the symmetry of w with respect to $x_2 = \pm x_1$. Summing up, using (5.3), we get $\sum_{p=p_k}^{k} J_{p,0}(w(\cdot + k\,\boldsymbol{e}_2)) = \sum_{p=p_k}^{k} J_{p,k}(w)\to 0$, so we can apply Lemma 3.3 and conclude that

$$\|w - v_k\|_{W^{1,2}(T_{k,k},\mathbb{R}^m)} \to 0\,. \tag{5.4}$$

Let us now consider, for every $k\in\mathbb{N}$, a different interpolation in the horizontal strip $\mathcal{S}_k$ between w and the periodic solution $v_k\in\mathcal{M}_0^{min}$ previously introduced:

$$\omega_k(x_1,x_2,y) = \begin{cases} w(x_1,x_2,y) & \text{if } 0\le x_1\le k \\ w(x_1,x_2,y)(k - x_1 + 1) + v_k(x_1,x_2,y)(x_1 - k) & \text{if } k < x_1 \le k+1 \\ v_k(x_1,x_2,y) & \text{if } x_1 > k+1 \\ \text{odd extended for } x_1<0 \end{cases}$$

Arguing as above $\|\omega_k - v_k\|_{W^{1,2}(T_{k,k},\mathbb{R}^m)} \le 2\|w - v_k\|_{W^{1,2}(T_{k,k},\mathbb{R}^m)}$, so that, defining $\omega_k^{\downarrow}(x) = \omega_k(x + k\boldsymbol{e}_2)$ in $\mathcal{S}_0$ we find $\|\omega_k^{\downarrow} - v_k\|_{W^{1,2}(T_{k,0},\mathbb{R}^m)} \to 0$ and hence, $J_{k,0}(\omega_k^{\downarrow}) \to 0$. Since $\omega_k^{\downarrow} \in \Gamma(v_k)$ and $v_k \in \mathcal{M}_0^{min}$ we obtain, reasoning as above,

$$c \le J(\omega_k^{\downarrow}) \le J_k(w) + 2J_{k,0}(\omega_k^{\downarrow}) = c + o(1)\,,$$

thus giving $J(\omega_k^{\downarrow}) \to c$.
We now prove that the sequence $(v_k)_k \in \mathcal{M}_0^{min}$ is indeed a (definitively) constant sequence, i.e. $v_k = \bar{v}$ for every k sufficiently large. Being $J(\omega_k^{\downarrow}) \to c$, we can assume $J(\omega_k^{\downarrow}) \le c + \tilde{\Lambda}(r_1)$ and since $\omega_k^{\downarrow} \in \Gamma(v_k)$ and $v_k \in \mathcal{M}_0^{min}$, we can apply Lemma 4.5 obtaining that

(i) $\|\omega_k^{\downarrow} - v_k\|_{W^{1,2}(T_{p,0},\mathbb{R}^m)} \le r_1$ for every $p \ge \tilde{\ell}(r_1)$;

(ii) $\sum_{p=\tilde{\ell}(\rho)}^{+\infty} J_{p,0}(\omega_k^{\downarrow}) \le 2\tilde{\Lambda}(r_1) < \frac{\lambda_0}{4}$;

As a consequence, by definition of $\omega_k^{\downarrow}$ and recalling that $\omega_k = w$ when $0 \le x_1 \le k$ we obtain $J_{p,k}(w) < \frac{\lambda_0}{4}$ and

$$\|w - v_k\|_{W^{1,2}(T_{p,k},\mathbb{R}^m)} \le r_1 < \frac{r_0}{4} \tag{5.5}$$

provided that $p_0 \le p \le k - 1$ where $p_0 = \tilde{\ell}(r_1)$. Consider now the *vertical rectangle* $[p_0, p_0 + 1] \times [p_0 + 1, +\infty) \times [0,1]^{n-2} = \cup_{k \ge p_0+1} T_{p_0,k}$. We have $J_{p_0,k}(w) \le \frac{\lambda_0}{4}$ for any k in the set of consecutive integers $\mathcal{I} = \{k \in \mathbb{Z} \mid k \ge p_0 + 1\}$, so that we can argue as in Lemma 3.3 and conclude that there exists $\bar{v} \in \mathcal{M}_0$ such that

$$\|w - \bar{v}\|_{W^{1,2}(T_{p_0,k},\mathbb{R}^m)} \le \frac{r_0}{4} \text{ for every } k \ge p_0 + 1\,. \tag{5.6}$$

Finally, recalling (2.4), since both (5.5) and (5.6) holds, we must have $\bar{v} = v_k \in \mathcal{M}_0^{min}$ for every $k \ge p_0 + 1$. In particular, (5.4) gives the claim in (5.2).
Moreover, we have proved that $(\omega_k^{\downarrow})_{k \ge p_0+1} \subset \Gamma(\bar{v})$ with $\bar{v} \in \mathcal{M}_0^{min}$ and since $J(\omega_k^{\downarrow}) \to c$, we can apply Lemma 4.10 to get that there exists $\bar{u} \in \mathcal{M}(\bar{v})$ for which, up to a subsequence,

$$\lim_{k\to+\infty} \|\omega_k^{\downarrow} - \bar{u}\|_{W^{1,2}(\mathcal{S}_0,\mathbb{R}^m)} = 0\,.$$

Hence we obtain that

$$\mathrm{dist}_{W^{1,2}(\mathcal{S}_0,\mathbb{R}^m)}(\omega_k^{\downarrow}, \mathcal{M}(\bar{v})) \to 0 \quad \text{as } k \to +\infty. \tag{5.7}$$

Finally, for every $u \in \mathcal{M}(\bar{v})$ we have

$$\begin{aligned}
\|w - u\|^2_{W^{1,2}(\mathcal{T}_k,\mathbb{R}^m)} &= 2\|w - u\|^2_{W^{1,2}(\cup_{p=0}^{k-1} T_{p,k},\mathbb{R}^m)} + \|w - u\|^2_{W^{1,2}(\tau_k,\mathbb{R}^m)} \\
&= \|\omega_k^{\downarrow} - u\|^2_{W^{1,2}(\mathcal{S}_0,\mathbb{R}^m)} - 2\|\omega_k^{\downarrow} - u\|^2_{W^{1,2}(T_{k,k},\mathbb{R}^m)} + \|w - u\|^2_{W^{1,2}(\tau_k,\mathbb{R}^m)} \\
&\le \|\omega_k^{\downarrow} - u\|^2_{W^{1,2}(\mathcal{S}_0,\mathbb{R}^m)} + \|w - u\|^2_{W^{1,2}(T_{k,k},\mathbb{R}^m)}\,.
\end{aligned}$$

Notice that since $u \in \Gamma(\bar{v})$ and using (5.2), we have $\|w - u\|^2_{W^{1,2}(T_{k,k},\mathbb{R}^m)} \le \|w - \bar{v}\|^2_{W^{1,2}(T_{k,k},\mathbb{R}^m)} + \|u - \bar{v}\|^2_{W^{1,2}(T_{k,k},\mathbb{R}^m)} \to 0$, as $k \to +\infty$. Hence, by (5.7), we conclude

$$\lim_{k\to+\infty} \mathrm{dist}_{W^{1,2}(\mathcal{T}_k,\mathbb{R}^m)}(w, \mathcal{M}(\bar{v})) = 0\,.$$

□

The previous lemma gives the asymptotic estimate in Theorem 1.1 since $\mathcal{R}_k \subset \mathcal{T}_k$.

We can conclude now the proof of Theorem 1.1 proving the sign property (i). By Lemma 2.2, for any periodic solution $v = (v_1, \ldots, v_m) \in \mathcal{M}_0^{min}$ we can define $v^a = (|v_1|, \ldots, |v_m|)$ belonging to $\mathcal{M}_0^{min}$ too, being $J_0(v^a) = J_0(v) = c_0$ easily verified. Now, by Theorem 4.6, there exists a heteroclinic solution $u = (u_1, \ldots, u_m) \in \mathcal{M}(v)$. We can define the function $u^a \in E^{odd}$, such that $u^a = (|u_1|, \ldots, |u_m|)$ when $x_1 \geq 0$, and verify that $u^a \in \mathcal{M}(v^a)$ being $J(u^a) = J(u) = c$.
Finally, for any $w = (w_1, \ldots, w_m) \in \widetilde{\mathcal{M}}$ we can find $v \in \mathcal{M}_0^{min}$ as in Lemma 5.5. Similarly as above, we can define $w^a \in \mathcal{E}$ such that $w^a = (|w_1|, \ldots, |w_m|)$ when $x_1 \geq 0$. Then, we can verify that $w^a \in \widetilde{\mathcal{M}}$ verifies Lemma 5.5 with the choice $v^a \in \mathcal{M}_0^{min}$. By reflecting w^a with rispect to the hyperplanes $x_2 = \pm x_1$, we obtain the saddle-type solution satisfying (i) in Theorem 1.1, thus completing the proof.

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Francesca Alessio
Dipartimento di Ingegneria Industriale e Scienze Matematiche,
Università Politecnica delle Marche,
Via Brecce Bianche 12,
60131 Ancona, Italy
`f.g.alessio@univpm.it`

Piero Montecchiari
Dipartimento di Ingegneria Civile e Architettura,
Università Politecnica delle Marche,
Via Brecce Bianche 12,
60131 Ancona, Italy
`p.montecchiari@univpm.it`

Andrea Sfecci
Dipartimento di Ingegneria Industriale e Scienze Matematiche,
Università Politecnica delle Marche,
Via Brecce Bianche 12,
60131 Ancona, Italy
`sfecci@dipmat.univpm.it`