# On a singular periodic Ambrosetti–Prodi problem

Alessandro Fonda and Andrea Sfecci

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**Abstract**

We investigate the possibility of extending a classical multiplicity result by Fabry, Mawhin and Nkashama to a periodic problem of Ambrosetti–Prodi type having a nonlinearity with possibly one or two singularities. In the second part of the paper we study the existence of periodic rotating solutions for radially symmetric systems with nonlinearities of the same type.

**Keywords:** periodic solutions, multiplicity results, lower and upper solutions, singularities, rotating solutions.
**MR (2000) Subject Classification**: 34C25.

## 1 Introduction

In 1972, Ambrosetti and Prodi [1] obtained a multiplicity result for the solutions of a Dirichlet problem associated to an elliptic equation, which can be said to have influenced the research in the field of boundary value problems up to the present days.

Let us recall the result of [1], as refined by Berger and Podolak in [3], by writing the Dirichlet problem as

$$\begin{cases} \Delta u + h(u) = s\varphi_1(x) + w(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

Here, $\Omega$ is a bounded domain in $\mathbb{R}^N$, while $\varphi_1(x)$ is the positive eigenfunction associated to the first eigenvalue $\lambda_1$ of the Laplacian, with Dirichlet boundary conditions, and $w(x)$ is a suitably smooth function. Assuming $h : \mathbb{R} \to \mathbb{R}$ to be twice continuously differentiable and strictly convex, with

$$0 < h'(-\infty) < \lambda_1 < h'(+\infty) < \lambda_2\,,$$

(where $\lambda_2$ is the second eigenvalue), they proved the existence of an $s_0 \in \mathbb{R}$ such that

- if $s < s_0$, there are no solutions,
- if $s = s_0$, there is exactly one solution,
- if $s > s_0$, there are exactly two solutions.

Since then, many variants and generalizations have been proposed, see e.g. [2, 4, 6, 15, 16, 17, 19, 20, 21, 22, 23, 26], a far from being exhaustive list. Remarkably, the name *Ambrosetti–Prodi problem* remained attached to all such situations when a multiplicity result structure as the one described above appears.

Searching for an analogue for the periodic problem, Fabry, Mawhin and Nkashama [7] considered in 1986 the second order differential equation

$$x'' + f(x)x' + h(t,x) = s\,. \tag{$E_s$}$$

(In this case, the Laplacian is replaced by a second derivative, and the first eigenvalue associated to the periodic problem is equal to zero.) They were able to prove the following Ambrosetti–Prodi type of result.

**Theorem 1.1 (Fabry–Mawhin–Nkashama)** *Assume $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to be continuous functions, with $T$-periodicity in the $t$ variable. If*

$$\lim_{|x|\to\infty} h(t,x) = +\infty\,, \qquad \textit{uniformly in } t \in [0,T]\,,$$

*then there exists an $s_0 \in \mathbb{R}$ such that*

- *if $s < s_0$, there are no $T$-periodic solutions,*
- *if $s = s_0$, there is at least one $T$-periodic solution,*
- *if $s > s_0$, there are at least two $T$-periodic solutions.*

We will take the above theorem as our starting point, and develop some possible generalizations. In the first part of the paper we focus our attention on the case when the nonlinearities in equation ($E_s$) are defined only for $x$ varying in an open interval $(a,b)$ of $\mathbb{R}$, with possibly one or two singularities. Here is our result, extending Theorem 1.1 to such a situation.

**Theorem 1.2** *Assume $f : (a,b) \to \mathbb{R}$ and $h : \mathbb{R} \times (a,b) \to \mathbb{R}$ to be continuous functions, with $T$-periodicity in the $t$ variable, such that*

$$\lim_{x\to a^+} h(t,x) = \lim_{x\to b^-} h(t,x) = +\infty\,, \quad \textit{uniformly in } t \in [0,T]. \tag{2}$$

*If $b = +\infty$, the same conclusion of Theorem 1.1 for equation ($E_s$) holds. On the other hand, if $b < +\infty$, the same is true assuming, in addition, that*

$$f(x) \geq 0 \quad \textit{and} \quad h(t,x) \geq h_m(x)\,, \quad \textit{for every} \quad x \in (a,b)\,,$$

*where* $h_m : (a,b) \to \mathbb{R}$ *is continuous and such that*

$$\int_c^b h_m(x)\,dx = +\infty\,, \tag{3}$$

*for some* $c \in (a,b)$.

A few comments on the above statement are in order. Notice that, in the case $(a,b) = \mathbb{R}$, Theorem 1.2 reduces to Theorem 1.1. If $b = +\infty$, no assumptions besides the continuity are required on the function $f$. When $b < +\infty$, the repulsive singularity at $x = b$ has to be sufficiently strong so to ensure that the solutions of $(E_s)$ cannot collide with it. On the contrary, it is remarkable that the attractive singularity at $x = a$ does not require an assumption of this type.

As a possible example of application of the above theorem, we propose the following physical model describing the dynamics of a charged particle in a periodically varying electric field. We consider a negatively charged particle, freely moving on a straight line between two fixed charged particles, one positive and the other one negative, with $T$-periodically varying (not vanishing) charges. We denote by $q^-$, $Q^+(t)$ and $Q^-(t)$ the electric charges (in absolute value), respectively. Let $x = x(t)$ be the position of the freely moving particle, and assume that the fixed charges are placed at $x = a$ and $x = b$ respectively, so that $a < x(t) < b$, for every $t$. We assume that the line of motion is confined between two capacitor plates, as in Figure 1. The equation of motion is then

$$x'' + k\left(\frac{Q^+(t)}{(x-a)^2} + \frac{Q^-(t)}{(x-b)^2}\right) = s\,, \tag{4}$$

with $k = q^-/4\pi\varepsilon m$ and $s = \sigma q^-/m\varepsilon$, where $m$ is the mass of the free charge, $\varepsilon$ is the dielectric permittivity and $\sigma$ is the surface charge density of the capacitor. As a consequence of Theorem 1.2, if $\sigma$ is large enough, equation (4) has at least two $T$-periodic solutions. A simple physical interpretation of this result can be easily given in the case when the electric charges $Q^+$ and $Q^-$ are constant in time: the strong constant force generated by the capacitor balances the attractive force exerted by $Q^+$, when the free particle $x(t)$ is near the position $x = a$, and the repulsive force exerted by $Q^-$, when $x(t)$ is near $x = b$. Hence there are two equilibria, one near $x = a$ (unstable) and the other one near $x = b$ (stable). When the electric charges $Q^+(t)$ and $Q^-(t)$ are not constant, but $T$-periodic in time, we have a perturbation of the previous situation, if $\sigma$ is large enough, and the equilibria we have found give rise to the two expected $T$-periodic solutions.

In the second part of the paper, we deal with a system of the type

$$\boldsymbol{x}'' = \big(-h(t,|\boldsymbol{x}|) + s\big)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,. \tag{$\mathcal{E}_s$}$$

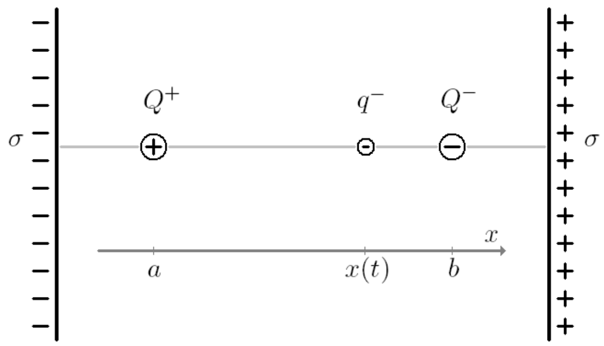


Figure 1: A charged particle $q^-$ in a capacitor of surface charge density $\sigma$, lying between the electric charges $Q^+$ and $Q^-$.

Here $\boldsymbol{x} = \boldsymbol{x}(t) \in \mathbb{R}^N$, and $|\cdot|$ denotes the euclidean norm. We will show that the same assumptions considered above on the nonlinearity $h : \mathbb{R} \times (a, b) \to \mathbb{R}$, with $a \geq 0$, lead to different types of periodic solutions: some of them will oscillate radially, being those provided by Theorem 1.2. However, new families of periodic solutions will arise, slowly rotating around the origin, completing a revolution in a period time which is a sufficiently large integer multiple of $T$.

Let us describe more precisely our result. Writing equation $(\mathcal{E}_s)$ in polar coordinates, we obtain the system

$$\begin{cases} \rho'' - \dfrac{\mu^2}{\rho^3} + h(t, \rho) = s\,, \\ \rho^2 \varphi' = \mu\,, \end{cases} \tag{$\mathcal{R}_s$}$$

where $\mu$ denotes the scalar angular momentum, which is known to be constant along the solutions. This fact is justified by the absence in $(\mathcal{E}_s)$ of the friction term related to the function $f$, which instead was included in the scalar equation $(E_s)$. We will only look for solutions with $\mu > 0$, since the ones with $\mu < 0$ can be obtained symmetrically. Notice that the solutions with $\mu = 0$ (hence with constant $\varphi$) oscillate radially and $\rho$ solves the scalar equation $(E_s)$, with $f = 0$. The rotating solutions we are looking for will be such that, for some positive integer $k$,

$$\rho(t + T) = \rho(t)\,, \qquad \varphi(t + kT) = \varphi(t) + 2\pi\,. \tag{5}$$

Notice that such solutions are $kT$-periodic, but their radial component is $T$-periodic. Let us state our result for the radially symmetric system $(\mathcal{R}_s)$.

**Theorem 1.3** *Let the same assumptions of Theorem 1.2 hold, with $f = 0$. Then, there exists an $s_0 \in \mathbb{R}$ such that, if $s > s_0$, system $(\mathcal{R}_s)$ has two families of rotating periodic solutions $(\rho_k, \varphi_k)$ with small angular momentum $\mu_k$.*

*More precisely, there exists a positive integer $k_s$ such that, for every integer $k \geq k_s$, there are two periodic solutions of $(\mathcal{R}_s)$ satisfying (5) and such that*

$$\lim_{k\to\infty} \mu_k = 0\,.$$

*In the case $b = +\infty$, if moreover*

$$\lim_{r\to+\infty} \frac{h(t,r)}{r} = 0\,, \tag{6}$$

*then $(\mathcal{R}_s)$ also admits rotating periodic solutions with large angular momentum: there exists a positive integer $\widehat{k}_s$ such that for every integer $k \geq \widehat{k}_s$ there is a periodic solution $(\rho_k, \varphi_k)$ of equation $(\mathcal{R}_s)$ satisfying (5) with the properties*

$$\lim_{k\to\infty} \min \rho_k = +\infty\,, \quad \lim_{k\to\infty} \frac{\min \rho_k}{\max \rho_k} = 1\,, \quad \lim_{k\to\infty} \mu_k = +\infty\,.$$

In the above statement, we may have several possible situations: in the case $b = +\infty$, if $a = 0$ we are in the classical case of a Keplerian-type system having only a singularity at the origin. Conversely, if $a > 0$ we have a *singular sphere* $\{\rho = a\}$ and the motion takes place outside of it. In the case $b < +\infty$, if $a = 0$ then we have the singularity at the origin and *one singular sphere* $\{\rho = b\}$ and the motion is confined inside of it. On the other hand, if $a > 0$, we have *two singular spheres*, $\{\rho = a\}$ and $\{\rho = b\}$, and the orbits lie in the annular region between them.

Equations of the type $(\mathcal{E}_s)$ have already been considered in the literature, taking into account several situations. E.g., systems with an attractive singularity of Keplerian type have been studied in [9, 12, 14]; the case of repulsive singularity has been treated in [10, 11, 13, 24]; bouncing solutions were found in [25].

The proof of Theorem 1.2 will be carried out in Section 2 by the use of upper and lower solutions and topological degree arguments, in the line of the proof given in [7]. Then, in Section 3, we will provide the proof of Theorem 1.3, adapting the techniques developed in [10, 12]. In Section 4 we end with some remarks on the possibility of weakening the assumptions, while still keeping the existence of at least one periodic solution, for large values of the parameter $s$.

## 2 Proof of Theorem 1.2

By hypothesis (2), we can define the real number

$$h_0 := \min\big\{h(t,x)\,:\,t\in[0,T],\,x\in(a,b)\big\}\,. \tag{7}$$

If $s < h_0$, we cannot have periodic solutions, since otherwise we would have $x''(t_0) < 0$ at any minimum point $t_0$. Let us fix an arbitrary $\xi \in (a, b)$ and define

$$h_1 := \max \big\{ h(t, \xi) \, : \, t \in [0, T] \big\} \geq h_0 . \tag{8}$$

Thanks to assumption (2), for every $s \in \mathbb{R}$ we can find an interval $[d_{1,s}, d_{2,s}] \subset (a, b)$ such that

$$h(t, x) > s \, , \text{ for every } (t, x) \in [0, T] \times \big( (a, d_{1,s}] \cup [d_{2,s}, b) \big) . \tag{9}$$

We can also assume that $s \mapsto d_{1,s}$ is decreasing and $s \mapsto d_{2,s}$ is increasing. Notice that $\alpha \equiv d_{1,s}$ and $\beta \equiv \xi$ are respectively a lower and an upper solution of $(E_s)$ with $\alpha < \beta$, for every $s > h_1$. Then, for every $s > h_1$ there exists a periodic solution $x$ of $(E_s)$ satisfying $d_{1,s} \leq x(t) \leq \xi$, for every $t \in [0, T]$. Hence, we can define

$$s_0 = \inf \big\{ s \in \mathbb{R} : \ (E_s) \text{ has a } T\text{-periodic solution} \big\} . \tag{10}$$

Notice that, by the previous reasoning, we have $h_0 \leq s_0 \leq h_1$.

The proof of the following lemma can be traced back to a pioneering paper by Lazer and Solimini [18].

**Lemma 2.1** *For every $s > s_0$, equation $(E_s)$ has a $T$-periodic solution.*

*Proof.* We fix $s > s_0$. There exists $\sigma \in [s_0, s)$ such that $(E_\sigma)$ has a $T$-periodic solution, which we denote by $x_\sigma$. It is easy to verify that $x_\sigma$ is an upper solution of $(E_s)$. Set $\alpha_\sigma \in \mathbb{R}$ such that $a < \alpha_\sigma < d_{1,s}$ and $\alpha_\sigma < \min x_\sigma$. Then $\alpha \equiv \alpha_\sigma$ is a lower solution of $(E_s)$, so that $(E_s)$ has a $T$-periodic solution $x_s$ satisfying $\alpha_\sigma \leq x_s(t) \leq x_\sigma(t)$, for every $t \in [0, T]$. The lemma is thus proved. ■

We now prove an priori estimate for all the possible $T$-periodic solutions of $(E_s)$, when $s$ varies in a compact interval.

**Lemma 2.2** *For every $\bar{s} > s_0$ there are two constants $\widehat{d}_{1,\bar{s}} < \widehat{d}_{2,\bar{s}}$ in $(a, b)$ such that every $T$-periodic solution $x$ of $(E_s)$ with $s \in [s_0, \bar{s}]$ must satisfy*

$$\widehat{d}_{1,\bar{s}} < x(t) < \widehat{d}_{2,\bar{s}} \, , \quad \textit{for every } t \in [0, T] . \tag{11}$$

*Proof.* Let $x$ be one such solution. We must have $\min x > d_{1,\bar{s}}$, otherwise we would have a negative second derivative at any minimum point. So, we can set $\widehat{d}_{1,\bar{s}} = d_{1,\bar{s}}$. Let us consider separately the cases $b = +\infty$ and $b < +\infty$.

*Case 1:* $b = +\infty$. Integrating equation $(E_s)$ we get

$$\frac{1}{T}\int_0^T h(t, x(t))\, dt = s\,.$$

Introducing the constants $d_{1,\bar{s}}, d_{2,\bar{s}}$ as in (9), there exists $t_0 \in [0, T]$ such that $x(t_0) \in (d_{1,\bar{s}}, d_{2,\bar{s}})$. Let us denote by $\bar{x}$ the mean value of $x$, i.e. $\bar{x} = \frac{1}{T}\int_0^T x(t)\, dt$, so that $\tilde{x}(t) = x(t) - \bar{x}$ has zero mean. Multiplying $(E_s)$ by $\tilde{x}$ we get

$$\begin{aligned}\|x'\|_2^2 = \int_0^T \tilde{x}(t) h(t, x(t))\, dt &= \int_0^T \tilde{x}(t)\big(h(t, x(t)) - h_0\big)\, dt \\ &\leq \|\tilde{x}\|_\infty \int_0^T \big(h(t, x(t)) - h_0\big)\, dt \leq \|\tilde{x}\|_\infty\, T(\bar{s} - h_0)\,.\end{aligned}$$

Let $t_1 \in [0, T]$ be such that $\tilde{x}(t_1) = 0$. Then, for every $t \in [t_1, t_1 + T]$,

$$|\tilde{x}(t)| \leq \left|\int_{t_1}^t \tilde{x}'(\tau)\, d\tau\right| \leq \int_0^T |x'(\tau)|\, d\tau \leq \sqrt{T}\, \|x'\|_2\,.$$

By the previous estimates, we get $\|x'\|_2 \leq T^{3/2}\,(\bar{s} - h_0)$, so that, for every $t \in [0, T]$,

$$x(t) = x(t_0) + \int_{t_0}^t x'(\tau)\, d\tau \leq x(t_0) + \sqrt{T}\|x'\|_2 < d_{2,\bar{s}} + T^2(\bar{s} - h_0)\,.$$

Hence, setting $\widehat{d}_{2,\bar{s}} = d_{2,\bar{s}} + T^2(\bar{s} - h_0)$, we have $x(t) < \widehat{d}_{2,\bar{s}}$, for every $t$. We have proved the a priori bound in the case $b = +\infty$.

*Case 2:* $b < +\infty$. We introduce the energy $E(x, y) = y^2/2 + H(x)$, where $H(x) = \int_c^x (h_m(\upsilon) - \bar{s})\, d\upsilon$. There exists a constant $H_0$ such that $H(\upsilon) \leq H_0$ for every $\upsilon \in [d_{1,\bar{s}}, d_{2,\bar{s}}]$. By (3), it is possible to find a $\widehat{d}_{2,\bar{s}} \in (d_{2,\bar{s}}, b)$ such that $H(\upsilon) > H_0$, for every $\upsilon \in [\widehat{d}_{2,\bar{s}}, b)$.

Assume $\max x > d_{2,\bar{s}}$. Then, by (9), there exist $t_1 < t_2$ such that $x'(t_1) = x'(t_2) = 0$, $x(t_2) = \max x$ and $x'(t) > 0$ for every $t \in (t_1, t_2)$, and $x(t_1) \in [d_{1,\bar{s}}, d_{2,\bar{s}}]$. A computation gives, for every $t \in (t_1, t_2)$,

$$\frac{d}{dt}E(x(t), x'(t)) = -x'(t)\big[f(x(t))x'(t) + h(t, x(t)) - s - h_m(x(t)) + \bar{s}\big] < 0\,.$$

(Here we have used the hypothesis $f \geq 0$.) Hence,

$$H_0 \geq H(x(t_1)) = E(x(t_1), x'(t_1)) \geq E(x(t_2), x'(t_2)) = H(x(t_2))\,,$$

thus giving us $x(t_2) < \widehat{d}_{2,\bar{s}}$. The a priori bound is thus proved, also in this case. ∎

We remark that it is possible to assume, without loss of generality, that $s \mapsto \widehat{d}_{1,s}$ is decreasing and $s \mapsto \widehat{d}_{2,s}$ is increasing. We now need an estimate on the derivative of the $T$-periodic solutions.

**Lemma 2.3** *For every $\bar{s} > s_0$, there exists a constant $D_{\bar{s}} > 0$ such that, if $x$ is a $T$-periodic solution of $(E_s)$ with $s \in [s_0, \bar{s}]$, then*

$$\|x'\|_\infty < D_{\bar{s}} .$$

*Proof.* Arguing by contradiction, we assume the existence of a sequence $(x_n)_n$ of $T$-periodic solutions of $(E_{s_n})$, with $s_n \in [s_0, \bar{s}]$, such that $M_n = \|x_n'\|_\infty > n$. There exist $t_{1,n} < t_{2,n}$ such that $x_n'(t_{1,n}) = 0$, $|x_n'(t_{2,n})| = M_n$ and $x_n'(t) \neq 0$ for every $t \in (t_{1,n}, t_{2,n})$. Without loss of generality we assume $x_n'(t) > 0$ for every $t \in (t_{1,n}, t_{2,n})$. We know from Lemma 2.2 that $\widehat{d}_{1,\bar{s}} < x_n(t) < \widehat{d}_{2,\bar{s}}$, for every $t \in [0,T]$. Hence, by $(E_s)$, we get the existence of a constant $C$ such that $x_n''(t) \le C\big(|x_n'(t)| + 1\big)$ for every $t$. We have

$$\left| \frac{x_n''(t) x_n'(t)}{x_n'(t) + 1} \right| \le C\, x_n'(t) ,$$

and, integrating in $(t_{1,n}, t_{2,n})$, we get

$$C\left(\widehat{d}_{2,\bar{s}} - \widehat{d}_{1,\bar{s}}\right) \ge C\big(x_n(t_{2,n}) - x_n(t_{1,n})\big) \ge \int_0^{M_n} \frac{w}{w+1}\, dw \stackrel{n\to\infty}{\longrightarrow} +\infty ,$$

a contradiction. ■

We now define our functional setting. Let $X = C([0,T])$ be the set of continuous functions, and let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to X$ be the operator defined as

$$\begin{aligned} &\mathcal{D}(\mathcal{L}) = \{x \in C^2([0,T]) \,:\, x(0) = x(T),\, x'(0) = x'(T)\} , \\ &\mathcal{L}x = x'' - x . \end{aligned}$$

Setting $Y = C^1([0,T])$, we define on

$$Y_{(a,b)} = \{x \in Y \,:\, a < x(t) < b ,\ \text{for every } t \in [0,T]\}$$

the Nemytskii operator $N_s : Y_{(a,b)} \to X$ as

$$(N_s\, x)(t) = -f(x(t))x'(t) - h(t, x(t)) + s - x(t) .$$

We thus have that $x$ is a $T$-periodic solution of $(E_s)$ if and only if it solves the equation

$$\mathcal{L}x = N_s\, x ,$$

with $x \in \mathcal{D}(\mathcal{L}) \cap Y_{(a,b)}$. Fix $\bar{s} > s_0$ and define the set

$$\Xi_{\bar{s}} = \Big\{ x \in Y : \widehat{d}_{1,\bar{s}} < x(t) < \widehat{d}_{2,\bar{s}} \ \text{ and } \ |x'(t)| < D_{\bar{s}} ,\ \text{for every } t \in [0,T] \Big\} .$$

By standard arguments, for every $s < \bar{s}$, the function $\Psi_s = \mathcal{L}^{-1} \circ N_s : \overline{\Xi}_{\bar{s}} \to Y$ is a completely continuous operator, and its fixed points are the $T$-periodic solutions of $(E_s)$. By Lemmas 2.2 and 2.3, we have that $0 \notin (I - \Psi_s)(\partial\, \Xi_{\bar{s}})$, so that the the degree of $I - \Psi_s$ on the open and bounded set $\Xi_{\bar{s}} \subset Y$ is well defined, for every $s \leq \bar{s}$. Recalling that for every $s < s_0$ there are no $T$-periodic solutions of $(E_s)$, using the homotopy invariance property of the degree, we have

$$d(I - \Psi_s, \Xi_{\bar{s}}) = 0\,, \text{ for every } s \leq \bar{s}\,.$$

Now, for every $\varepsilon > 0$, consider a $T$-periodic solution $\beta_\varepsilon$ of equation $(E_{s_0+\varepsilon})$. So, $\alpha_\varepsilon \equiv \widehat{d}_{1,\bar{s}}$ and $\beta_\varepsilon$ are respectively a lower and an upper solution of $(E_s)$, when $s_0 + \varepsilon < s \leq \bar{s}$. Set

$$\Omega^1_\varepsilon = \left\{ x \in Y \,:\, \alpha_\varepsilon < x(t) < \beta_\varepsilon(t) \;\text{ and }\; |x'(t)| < D_{\bar{s}}\,,\; \text{for every } t \in [0,T] \right\},$$

a subset of $\Xi_{\bar{s}}$, by Lemmas 2.2 and 2.3. We now prove that there are no $T$-periodic solutions of $(E_s)$ belonging to $\partial\Omega^1_\varepsilon$, if $s \in (s_0 + \varepsilon, \bar{s}]$. Let $x$ be a $T$-periodic solution of $(E_s)$ such that $\alpha_\varepsilon \leq x(t) \leq \beta_\varepsilon(t)$, for every $t \in [0,T]$. Arguing as above, we see that such a solution cannot have $\alpha_\varepsilon$ as a minimum. Conversely, suppose that there exists a $\tau \in [0,T]$ such that $x(\tau) - \beta_\varepsilon(\tau) = 0$. Then, $\tau$ is a point of maximum for $x(t) - \beta_\varepsilon(t)$, hence $x'(\tau) - \beta'_\varepsilon(\tau) = 0$, and we have

$$\begin{aligned} x''(\tau) - \beta''_\varepsilon(\tau) = -f(x(\tau))x'(\tau) - h(\tau, x(\tau)) + s + f(\beta_\varepsilon(\tau))\beta'_\varepsilon(\tau) \\ + h(\tau, \beta_\varepsilon(\tau)) - s_0 - \varepsilon = s - s_0 - \varepsilon > 0\,, \end{aligned}$$

leading to a contradiction. Hence, by a standard result in lower and upper solution theory (see, e.g., [5]),

$$d(I - \Psi_s, \Omega^1_\varepsilon) = 1\,, \text{ for every } s \in (s_0 + \varepsilon, \bar{s}]\,. \tag{12}$$

We now define $\Omega^2_\varepsilon = \Xi_{\bar{s}} \setminus \overline{\Omega^1_\varepsilon}$. By the additivity property of the degree,

$$d(I - \Psi_s, \Omega^2_\varepsilon) = -1\,, \text{ for every } s \in (s_0 + \varepsilon, \bar{s}]\,. \tag{13}$$

Hence, since the choice of $\varepsilon$ is arbitrary, for every $s \in (s_0, \bar{s}]$ there are at least two $T$-periodic solutions of $(E_s)$, one in $\Omega^1_\varepsilon$ and the second one in $\Omega^2_\varepsilon$, simply choosing $\varepsilon < s - s_0$. Since we can consider $\bar{s} > s_0$ arbitrarily large, we have thus proved that there exist two $T$-periodic solutions of $(E_s)$, for every $s > s_0$.

In order to prove the existence of at least one periodic solution of $(E_{s_0})$, we consider a strictly decreasing sequence $(s_n)_n$ with $\lim s_n = s_0$. For every $n$, let $x_n$ be a solution of $(E_{s_n})$. By Lemma 2.2, we have that $(x_n)_n$ is contained in $\Xi_{\bar{s}}$. Using Lemma 2.3 and the fact that $x_n$ solves the differential equation $(E_{s_n})$, we have that $(x_n)_n$ is bounded in $C^2([0,T])$, so that, by the Ascoli-Arzelà Theorem, it $C^1$-converges up to subsequences to some $x \in \overline{\Xi}_{\bar{s}}$. Since $x_n = \Psi_{s_n}(x_n)$, passing to the limit we obtain that $x = \Psi_{s_0}(x)$, so that $x$ solves $(E_{s_0})$. The proof of Theorem 1.2 is thus completed.

## 3 Proof of Theorem 1.3

Setting $X = C([0,T])$ and

$$X_{(a,b)} = \{\rho \in X \,:\, a < \rho(t) < b\,,\ \text{for every } t \in [0,T]\}\,,$$

the Nemytskii operator $N_{s,\mu} : X_{(a,b)} \to X$ can now be defined as

$$(N_{s,\mu}\,\rho)(t) := \frac{\mu^2}{\rho^3(t)} - h(t,\rho(t)) + s - \rho(t)\,.$$

Let $\Omega$ be an open bounded subset of $X_{(a,b)}$ such that $\overline{\Omega} \subset X_{(a,b)}$. The operator $\Psi_{s,\mu} = \mathcal{L}^{-1} \circ N_{s,\mu} : \overline{\Omega} \to X$ is completely continuous and its fixed points correspond to $T$-periodic solutions of the first equation in $(\mathcal{R}_s)$.

The following theorem is a variant of [12, Theorem 2].

**Theorem 3.1** *Assume that there are no fixed points of $\Psi_{s,0}$ on $\partial\Omega$, and that $d(I - \Psi_{s,0}, \Omega) \neq 0$. Then, there exists a $\bar{k} \geq 1$ such that, for every integer $k \geq \bar{k}$, equation $(\mathcal{R}_s)$ has a $kT$-periodic solution $(\rho_k, \varphi_k)$ satisfying (5). Moreover, $\rho_k$ belongs to $\Omega$ and, if $\mu_k$ denotes the associated angular momentum, then*

$$\lim_{k\to\infty} \mu_k = 0\,.$$

The proof of Theorem 3.1 is completely analogous of the one provided in [12]. It can also be carried out by suitably modifying the nonlinearity $h$, in the following way. Take an interval $[c,d] \subset (a,b)$ such that $\overline{\Omega} \subset X_{(c,d)}$, and a function $\widetilde{h} : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ such that $\widetilde{h} = h$ on $\mathbb{R} \times (c,d)$. Replacing $h$ with $\widetilde{h}$ in $(\mathcal{R}_s)$ we are brought back to the setting of a Newtonian system already considered in [12, Theorem 2], and the result follows.

Going back to the first equation in $(\mathcal{R}_s)$, we first study the situation when $\mu = 0$. Following the proof of the first part of Theorem 1.3, we fix $\bar{s} > s_0$ and define

$$\Xi_{\bar{s}} = \big\{\rho \in X : \widehat{d}_{1,\bar{s}} < \rho(t) < \widehat{d}_{2,\bar{s}}\,,\ \text{for every } t \in [0,T]\big\}.$$

Taking $s \in (s_0, \bar{s}]$, we can choose $\varepsilon < \bar{s} - s_0$ and define

$$\Omega^1_\varepsilon = \big\{\rho \in X \,:\, \alpha_\varepsilon < \rho(t) < \beta_\varepsilon(t)\,,\ \text{for every } t \in [0,T]\big\}\,,$$

a subset of $\Xi_{\bar{s}}$, where we have used the notation of the previous section for $\alpha_\varepsilon$ and $\beta_\varepsilon(t)$. Finally, we set $\Omega^2_\varepsilon = \Xi_{\bar{s}} \setminus \overline{\Omega^1_\varepsilon}$. We thus obtain the analogues of formulas (12) and (13), i.e.,

$$d(I - \Psi_{s,0}, \Omega^1_\varepsilon) = 1\,, \qquad d(I - \Psi_{s,0}, \Omega^2_\varepsilon) = -1\,,$$

for every $s \in (s_0 + \varepsilon, \bar{s}]$, and we can apply Theorem 3.1 with $\Omega = \Omega^1_\varepsilon$ and $\Omega = \Omega^2_\varepsilon$, thus finding the two required families of periodic solutions with a small angular momentum.

When $b = +\infty$, the proof of the second part of the statement follows directly from [10, Theorem 1.2], after suitably modifying the function $h : \mathbb{R} \times (a, +\infty) \to \mathbb{R}$ to some function $\widehat{h} : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ such that $\widehat{h} = h$ on $\mathbb{R} \times (c, +\infty)$, for some $c > a$.

## 4 Final remarks

Adapting the proof of Theorem 1.2, it is easy to see that the following result also holds.

**Corollary 4.1** *Assume*

$$\lim_{x \to a^+} h(t, x) = +\infty \,, \qquad \textit{uniformly in } t \in [0, T]. \tag{14}$$

*Then, there exists at least one $T$-periodic solution of equation $(E_s)$, provided that $s$ is sufficiently large.*

Indeed, in such a situation it is possible to find a lower and an upper solution of $(E_s)$, for large values of $s$. We emphasize that, being now $h$ not necessarily bounded from below, the value $s_0$ introduced in (10) can be equal to $-\infty$.

As an example of application of Corollary 4.1, a physical model similar to the one described in Figure 1 can be considered, by dropping the charge $Q^-$, or replacing it with a positive one. In the former case $s_0 \in \mathbb{R}$, and in the latter $s_0 = -\infty$.

At this point, arguing as in the proof of Theorem 1.3, it is possible to find at least one family of rotating solutions for the radially symmetric system $(\mathcal{E}_s)$, with a small angular momentum.

As a final remark on the bibliography, we notice that the case when $Q^+$ is replaced by a negative charge has been considered in [8], and the corresponding result for rotating solutions has been discussed in [13], assuming some symmetry conditions on the nonlinearity.

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Authors' addresses:


Alessandro Fonda
Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
P.le Europa, 1
I-34127 Trieste, Italy
e-mail: a.fonda@units.it

Andrea Sfecci
Università Politecnica delle Marche
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Via Brecce Bianche, 12
I-60131 Ancona, Italy
e-mail: sfecci@dipmat.univpm.it