Multiplicity of radial ground states for the scalar curvature equation without reciprocal symmetry

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Abstract

We study existence and multiplicity of positive ground states for the scalar curvature equation

$$\Delta u + K(|x|)\, u^{\frac{n+2}{n-2}} = 0, \quad x \in \mathbb{R}^n\,, \quad n > 2,$$

when the function $K : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded above and below by two positive constants, i.e. $0 < \underline{K} \le K(r) \le \overline{K}$ for every $r > 0$, it is decreasing in $(0, \mathcal{R})$ and increasing in $(\mathcal{R}, +\infty)$ for a certain $\mathcal{R} > 0$.
We recall that in this case ground states have to be radial, so the problem is reduced to an ODE and, then, to a dynamical system via Fowler transformation.

We provide a smallness non perturbative (i.e. computable) condition on the ratio $\overline{K}/\underline{K}$ which guarantees the existence of a large number of ground states with fast decay, i.e. such that $u(|x|) \sim |x|^{2-n}$ as $|x| \to +\infty$, which are of bubble-tower type.

We emphasize that if $K(r)$ has a unique critical point and it is a maximum the radial ground state with fast decay, if it exists, is unique.

Key Words: scalar curvature equation, ground states, Fowler transformation, invariant manifold, shooting method, bubble tower solutions, phase plane analysis, multiplicity results.
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1 Introduction

In this paper we study the scalar curvature equation

$$\Delta u + K(|x|)\, u^{\frac{n+2}{n-2}} = 0, \quad x \in \mathbb{R}^n\,, \quad n > 2, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $n > 2$, $K \in C^1$ and $0 < \underline{K} \le K(|x|) \le \overline{K}$ whenever $|x| > 0$ for suitable positive constants $0 < \underline{K} < \overline{K}$. In particular, we will focus our attention on radially symmetric positive solutions with fast decay. Radial solutions of equation (1.1) solve

$$(u'\, r^{n-1})' + K(r)\, r^{n-1}\, u^{\frac{n+2}{n-2}} = 0\,, \qquad r \in (0, \infty). \tag{1.2}$$

We are interested in studying multiplicity of grounds states u of (1.2) having fast decay, where with *ground state* (GS) we mean a positive regular solution u of (1.2) defined for any $r \ge 0$ and for *fast decay* we mean that $u(r)r^{2-n}$ has positive finite limit as $r \to +\infty$.

The existence of GS with fast decay has been the subject of many papers for its intrinsic mathematical interest, but also for the relevant applications it finds in differential geometry, astrophysics and quantum mechanics, see [16], and reference therein for more details on applications. Extensive studies have been developed since the pioneering work [33] by Ni.

It was early realized that if K is monotone (but not constant) no GS with fast decay can exist, due to the Pohozaev obstruction, see e.g. [17]. In the 90s several different conditions ensuring existence of GS with fast decay have been obtained when K has a positive critical point, see the introduction in [16] for a more detailed discussion of this point.

Let us focus on the case where $K(r)$ has a unique critical point, say $K(\mathcal{R})$: the structure of positive solutions of (1.1) changes drastically whether $K(\mathcal{R})$ is a maximum or a minimum. First of all, if $K(\mathcal{R})$ is a maximum, we can have existence [34] and multiplicity [36] of non-radial GS with fast decay; while, if $K(\mathcal{R})$ is a minimum, Bianchi showed in [6, Theorem 2] that all the positive solutions have to be radial.

Furthermore, if $K(\mathcal{R})$ is a maximum, it is possible to classify completely positive radial solutions to get uniqueness of the radial GS with fast decay [31, 38, 39] and to obtain a structure result for radial solutions both regular and singular, also when they are sign changing [15], and even in the p-Laplace context [20–22, 24]. It is also worth noticing that multiplicity results can be attained in presence of multiple critical points of K, see e.g. [3, 4, 9, 24, 29, 37].

When $K(\mathcal{R})$ is a minimum, the situation is more difficult and, concerning GS with fast decay, it ranges from non-existence [7] to existence and multiplicity results, [7, 8] and [1, 10, 16, 18, 32], respectively.

Despite of the fact that the search of GS with fast decay has received great attention, much less results are available concerning multiplicity when K is bounded and has a unique critical point (a minimum, otherwise we have uniqueness [31]). In fact, as far as we are aware, [10, 16, 18, 32] is the complete list of papers on this topic. No results are available in a p-Laplace context; furthermore, this paper and [16] are the only non-perturbative contributions.

Remark 1.1. *Rescaling u, we can assume that the function K is such that*

$$K(|x|) = 1 + \varepsilon k(|x|)\,, \qquad 0 \le k(|x|) \le 1, \tag{1.3}$$

for every $x > 0$, see [16, Remark 1.3].

In [10] and [16], the simplifying condition that K is reciprocally symmetric is required, i.e. it is assumed

($\mathbf{K_0}$) $K(r) = K\left(\frac{1}{r}\right)$ for any $0 < r \le 1$;

($\mathbf{K_1}$) $K'(r) \le 0$ for any $r \in (0, 1]$, but $K'(r) \not\equiv 0$;

($\mathbf{K_2}$) $K(r) = K(0) - Ar^l + h(r)\,,$ where
$$A > 0, \qquad 0 < l < \frac{n-2}{2}, \qquad \lim_{r\to 0} |h(r)|r^{-l} + |h'(r)|r^{-l+1} = 0.$$

Let us recall here their main theorem.

Theorem 1.2 ([10,16]). *Assume that K satisfies* (1.3) *and* **($\mathbf{K_0}$)-($\mathbf{K_1}$)-($\mathbf{K_2}$)**, *then for any $\ell \in \mathbb{N}$ there exists a $\eta_\ell > 0$ such that for every $\varepsilon \in (0, \eta_\ell)$ equation* (1.2) *admits at least ℓ GS with fast decay $u_1, \ldots, u_\ell$, where the function $u_j(r)r^{\frac{n-2}{2}}$ has j local maxima and $(j-1)$ local minima.*

In [16] the authors, inspired by the previous work [10] of Chen and Lin, explicitly compute the constants η_ℓ, extending the perturbative result in [10] to a non-perturbative situation. Let us recall here their values for the reader's convenience.

Theorem 1.3 ([16]). *All the constants η_ℓ in Theorem 1.2 can be explicitly computed. In particular, we find the following:*

n	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8
3	2	0.910	0.584	0.429	0.339	0.280	0.238	0.207
4	1	0.5	0.333	0.25	0.2	0.166	0.142	0.125
5	0.666	0.347	0.235	0.178	0.143	0.119	0.103	0.090
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

(1.4)

Moreover, the explicit expression of the first two constants is

$$\eta_1 = \frac{2}{n-2}, \quad \eta_2 = \frac{2}{n}\left[\left(\frac{n}{n-2}\right)^{\frac{n-2}{2}} - 1\right]^{-1}. \tag{1.5}$$

Finally, if the dimension is $n = 4$, we have $\eta_\ell = \dfrac{1}{\ell}$ for every positive integer ℓ.

Remark 1.4. *The value of η_ℓ found in Theorem 1.3 is not optimal, i.e. we cannot say that if $\varepsilon > \eta_\ell$ we do not have ℓ G.S with fast decay.*

In the present paper we are going to study the structure of GS with fast decay when K is not reciprocally symmetric, i.e. when we remove assumption **($\mathbf{K_0}$)**. In particular, we assume

($\mathbf{H_1}$) $K'(r) \le 0$ for any $r \in (0, \mathcal{R}]$, but $K'(r) \not\equiv 0$;
$K'(r) \ge 0$ for any $r \in [\mathcal{R}, +\infty)$, but $K'(r) \not\equiv 0$;

($\mathbf{H_2^0}$) $K(r) = K_0 - a_0 r^{l_0} + h_0(r),$ where
$$a_0 > 0, \qquad 0 < l_0 < \frac{n-2}{2}, \qquad \lim_{r\to 0} |h_0(r)|r^{-l_0} + |h_0'(r)|r^{-l_0+1} = 0;$$

$(\mathbf{H_2^\infty})$ $K(r) = K_\infty - a_\infty r^{-l_\infty} + h_\infty(r)$, where

$$a_\infty > 0, \quad 0 < l_\infty < \frac{n-2}{2}, \quad \lim_{r\to+\infty} |h_\infty(r)| r^{l_\infty} + |h'_\infty(r)| r^{l_\infty+1} = 0.$$

We set $\mathcal{T} := \ln(\mathcal{R})$. This notation will be followed in the whole paper.

Remark 1.5. *According to [5, Theorem 1] and [6, Theorem 2], we know that assumption* $(\mathbf{H_1})$ *guarantees that each solution of* (1.1) *is radially symmetric about the origin. Thus, the search for GS of* (1.1) *is reduced to the analysis of* (1.2).

Theorem 1.6. *Assume that K satisfies* (1.3) *and* $(\mathbf{H_1})$-$(\mathbf{H_2^0})$-$(\mathbf{H_2^\infty})$*, then for any $\ell \in \mathbb{N}$ there exists $\varepsilon_\ell := \eta_{2\ell} > 0$ such that for every $\varepsilon \in (0, \varepsilon_\ell)$ equation* (1.2) *admits at least ℓ GS with fast decay $u_1, \ldots, u_\ell$, where the function $u_j(r) r^{\frac{n-2}{2}}$ has j local maxima and $(j-1)$ local minima, when $r > 0$.*

We emphasize that the constants ε_ℓ in the statement of our main theorem are the same constants obtained in [16] and can be explicitly computed, in particular we find $\varepsilon_\ell := \eta_{2\ell}$. Thus, when $0 < \varepsilon < \varepsilon_\ell$ we get the existence of at least 2ℓ GS with fast decay if k is reciprocally symmetric (via Theorem 1.2), while we find just ℓ GS if such a symmetry is lost (via Theorem 1.6).
Hence, if k is not symmetric we find the following table:

n	ε_1	ε_2	ε_3	ε_4
3	0.910	0.429	0.280	0.207
4	0.5	0.25	0.166	0.125
5	0.347	0.178	0.119	0.090
6	0.266	0.138	0.093	0.070

(1.6)

Moreover, $\varepsilon_\ell = \dfrac{1}{2\ell}$ if the dimension is $n = 4$, and $\varepsilon_1 = \dfrac{2}{n}\left[\left(\dfrac{n}{n-2}\right)^{\frac{n-2}{2}} - 1\right]^{-1}$.

In fact, Lin and Liu in [32] obtained an analogous multiplicity result in the absence of symmetry, but again in a perturbative setting. Namely, in [32, Theorem 1.3] they require $K_0 = K_\infty = 1$ in $(\mathbf{H_2^0})$, $(\mathbf{H_2^\infty})$, and for any $\ell \in \mathbb{N}$ they obtain a small but computable $\bar{l}_\ell \searrow 0$ ($\bar{l}_\ell := \frac{n-2}{(n+2)(\ell-1)}$) and a small (unprecisely small) $\varepsilon_\ell > 0$ such that (1.2) admits at least ℓ GS with fast decay whenever $0 < l_0, l_\infty < \bar{l}_\ell$ and $0 < \varepsilon < \varepsilon_\ell$. So, we improve the result by Lin and Liu by allowing $K_0 \neq K_\infty$, by giving a computable lower bound for the size of ε_ℓ and by considerably relaxing the smallness condition on l_0 and l_∞. Notice in fact that $\bar{l}_\ell < \frac{n-2}{2}$ for any $\ell \geq 2$. Furthermore, we believe that the condition $l_0, l_\infty < \frac{n-2}{2}$ in $(\mathbf{H_2^0}), (\mathbf{H_2^\infty})$ is optimal, see Remark 2.8.

Along the same line, the paper [18] provides multiplicity results for equation (1.2) in presence of a singular perturbation $K(r)$ of the form

$$K(r) = k(r^\varepsilon). \tag{1.7}$$

This peculiar perturbation amounts to ask for k to vary slowly; nevertheless it allows to get multiple GS with fast decay under weaker conditions: it is sufficient to assume that $k(r)$ in (1.7) is strictly positive, bounded and admits at least a positive minimum to reach the same conclusion as in Theorem 1.2. However, the result is of perturbative nature and we do not have any clue about how small ε_ℓ should be.

We emphasize that all the GS found in [10, 16, 18, 32] and in Theorem 1.6 are, in fact, bubble tower solutions, i.e. they are well approximated, as $\varepsilon \to 0$, by the sum of j explicitly known solutions of equation (1.2) with $\varepsilon = 0$, for $1 \le j \le \ell$, see e.g. [32] for more details.

To conclude this quick review of the existing literature we recall [36], where Wei and Yan prove that if $K(|x|)$ has a positive maximum, there are infinitely many non-radial GS (and a unique radial GS). This result, together with [10,16, 18,32] and the present article, suggests that the bubble tower phenomenon arises in presence of a critical point of $K(|x|)$, and it is made up by radial solutions if the critical point is a minimum and by non-radial ones if it is a maximum.

As in [16], Theorem 1.6 can be trivially generalized to embrace the slightly more general case of

$$(u'r^{n-1})' + r^{n-1+\sigma}[1+\varepsilon k(|x|)]u^{q(\sigma)-1} = 0, \quad 0 \le k(|x|) \le 1, \tag{1.8}$$

where $q(\sigma) = 2\,\frac{n+\sigma}{n-2}$, $\sigma > -2$. In [16, Corollary 1.4] the authors prove the existence of a positive computable value η_ℓ^σ, depending on σ, such that for every $\varepsilon \in (0, \eta_\ell^\sigma)$ equation (1.8) admits at least ℓ GS with fast decay, under the symmetric assumption $(\mathbf{K_0})$. This result admits the following extension to the non-symmetric setting.

Corollary 1.7. *Assume that $K = 1+\varepsilon k$ satisfies $(\mathbf{H_1})$-$(\mathbf{H_2^0})$-$(\mathbf{H_2^\infty})$, then for any $\ell \in \mathbb{N}$ there exists $\varepsilon_\ell^\sigma := \eta_{2\ell}^\sigma > 0$ such that for every $\varepsilon \in (0, \varepsilon_\ell^\sigma)$ equation (1.8) admits at least ℓ GS with fast decay $u_1, \dots, u_\ell$, where the function $u_j(r) r^{\frac{n-2}{2}}$ has j local maxima and $(j-1)$ local minima in $(0, +\infty)$. In particular, $\varepsilon_\ell^0 = \varepsilon_\ell$, and*

$$\varepsilon_1^\sigma = \frac{q-2}{q}\left[\left(\frac{q}{2}\right)^{\frac{2}{q-2}} - 1\right]^{-1} = \frac{2+\sigma}{n+\sigma}\left[\left(\frac{n+\sigma}{n-2}\right)^{\frac{n-2}{2+\sigma}} - 1\right]^{-1}. \tag{1.9}$$

Note that the table (1.6) refers to the specific case $\sigma = 0$. So, it should be slightly modified according to the construction in [16], in order to include the σ-dependence.
Corollary 1.7 allows us to deduce multiplicity of *radial* GS with fast decay for

$$\Delta u + r^\sigma[1+\varepsilon k(|x|)]u^{q(\sigma)-1} = 0, \tag{1.10}$$

since the solutions to equation (1.8) correspond to the *radial* solutions of (1.10). We emphasize that the GS of (1.10) need not be radial, since it is not possible to apply directly [6, Theorem 2] to equation (1.10).

The proofs of Theorem 1.6 and Corollary 1.7 are obtained by transforming (1.8), via Fowler transformation, into the non-autonomous dynamical system (2.2) introduced below. Regular solutions will be part of the unstable leaf $W^u(\mathcal{T})$, while fast decay solutions will be part of the stable leaf $W^s(\mathcal{T})$: the existence of a GS with fast decay is then converted in the search for an intersection between $W^u(\mathcal{T})$ and $W^s(\mathcal{T})$, or, equivalently, in the search for a homoclinic trajectory of (2.2). The Fowler transformation allows to study elliptic equations by adopting techniques borrowed from dynamical systems theory such as phase plane analysis, invariant manifolds, homotopies and some ideas inspired by Melnikov theory, providing a new geometrical perspective on the given equations. This approach started by Fowler in [19], developed and deepened by Jones et al. in [13, 30] and Johnson et al. in [26–29], have turned out to be very fruitful in several different contexts.

The paper is organized as follows. In §2 we introduce the Fowler transformation to convert equation (1.8) into the dynamical system (2.2), we review some basic properties concerning the new formulation of the problem, and we recall the crucial construction of a guiding curve for the solutions of (2.2) conceived in [16]. In §3 we use a topological argument to get information on the unstable leaf $W^u(\mathcal{T})$. In §4 we use the Kelvin inversion to translate information on the unstable leaf $W^u(\cdot)$ into information on the stable leaf $W^s(\cdot)$. Finally, in §5 we prove the existence of multiple intersections between $W^u(\mathcal{T})$ and $W^s(\mathcal{T})$, thus proving Theorem 1.6 and Corollary 1.7.

2 Preliminary results borrowed from [16]

In this Section we recall the main tools introduced in [16].

The Fowler transformation

$$\begin{array}{ll} x(t) = u(r)r^{\alpha}, & y(t) = \alpha u(r)r^{\alpha} + u'(r)r^{\alpha+1}\,, \\ \alpha = \frac{n-2}{2}, & r = e^t, \qquad \mathcal{K}(t) = K(e^t) \end{array} \tag{2.1}$$

permits us to pass from (1.8) to the following two-dimensional dynamical system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\mathcal{K}(t)\, x^{q-1} \end{pmatrix}, \tag{2.2}$$

where " $\cdot$ " denotes the differentiation with respect to t, and

$$q = q(\sigma) = 2\,\frac{n+\sigma}{n-2} > 2.$$

We recall that if $\sigma = 0$, i.e. if we consider (1.2), then $q = q(0) = \frac{2n}{n-2}$.

Let $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q})$ be the trajectory of (2.2) which is in $\boldsymbol{Q} \in \mathbb{R}^2$ at $t = \tau$. We denote by $u(r; d)$ the regular solution of (1.8) satisfying $u(0; d) = d > 0$, and by $\boldsymbol{\phi}(t; d) = (x(t; d), y(t; d))$ the corresponding trajectory of (2.2). From a standard application of L'Hôpital rule, we find that $u'(0; d) = 0$, whenever $\sigma > -1$; thus $u(x) = u(|x|; d)$ is smooth as solution of (1.10).

Similarly, we denote by $v(r; c)$ the fast decay solution of (1.8) such that $\lim_{r\to\infty} v(r; c)r^{n-2} = c > 0$, and by $\boldsymbol{\psi}(t; c)$ the corresponding trajectory of (2.2).

Remark 2.1. *In the whole paper we assume that the function $K \in C^1(0, \infty)$ has positive finite limit both as $r \to 0$ and as $r \to +\infty$, and that there exists $\varpi > 0$ such that*

$$\lim_{r\to 0} K'(r)r^{1-\varpi} = 0 \quad \text{and} \quad \lim_{r\to\infty} K'(r)r^{1+\varpi} = 0.$$

These assumptions (cf. [23]) are enough to guarantee that the function $\mathcal{K}(t)$ is uniformly continuous in $\mathbb{R}$ and to deduce the existence of the stable and unstable manifolds, see e.g. [14, §13.4] or [29].

Notice that if the hypotheses $(\mathbf{H_2^0})$ *and* $(\mathbf{H_2^\infty})$ *hold true, the assumptions of this Remark are satisfied by choosing* $\varpi = \frac{1}{2}\min\{l_0, l_\infty\}$.

Let us briefly recall some contents of [16, §2].

Proposition 2.2. *Assume that the hypotheses of Remark* 2.1 *are satisfied.*

1. *The origin is a saddle-type critical point for* (2.2) *and admits unstable and stable leaves* $\hat{W}^u(\tau)$ *and* $\hat{W}^s(\tau)$*, which are* C^1 *immersed manifolds. The origin splits* $\hat{W}^u(\tau)$ *and* $\hat{W}^s(\tau)$ *in two connected components: we denote respectively by* $W^u(\tau)$ *and* $W^s(\tau)$ *the ones which leave the origin and enter* $x > 0$*, i.e.*

$$\begin{aligned} W^u(\tau) :=\ & \{\boldsymbol{Q} \mid \lim_{t\to-\infty} \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) = (0,0),\ x(t;\tau,\boldsymbol{Q}) > 0 \text{ when } t \ll 0\}, \\ W^s(\tau) :=\ & \{\boldsymbol{Q} \mid \lim_{t\to+\infty} \boldsymbol{\phi}(t;\tau,\boldsymbol{Q}) = (0,0),\ x(t;\tau,\boldsymbol{Q}) > 0 \text{ when } t \gg 0\}. \end{aligned} \tag{2.3}$$

   $W^u(\tau)$ *and* $W^s(\tau)$ *are* C^1 *immersed one-dimensional manifolds for any* $\tau \in \mathbb{R}$.

2. *For every* $d > 0$, $u(r;d)$ *is a regular solution if and only if the corresponding trajectory* $\boldsymbol{\phi}(\cdot;d)$ *is such that* $\boldsymbol{\phi}(\tau;d) \in W^u(\tau)$ *for every* $\tau \in \mathbb{R}$. *Moreover,* $W^u(\tau)$ *is tangent in the origin to the line* $y = \alpha x$*, for any* $\tau \in \mathbb{R}$.

3. *For every* $c > 0$, $v(r;c)$ *is a fast decay solution if and only if the corresponding trajectory* $\boldsymbol{\psi}(\cdot;c)$ *is such that* $\boldsymbol{\psi}(\tau;c) \in W^s(\tau)$ *for every* $\tau \in \mathbb{R}$. *Moreover,* $W^s(\tau)$ *is tangent in the origin to the line* $y = -\alpha x$*, for any* $\tau \in \mathbb{R}$.

4. $W^u(\tau)$ *and* $W^s(\tau)$ *depend smoothly on* τ*; i.e. let* L *be a segment which intersects* $W^u(\tau_0)$ *– or* $W^s(\tau_0)$ *– transversally in a point* $\boldsymbol{Q}(\tau_0)$*, then there is a neighborhood* I *of* τ_0 *such that* $W^u(\tau)$ *– or* $W^s(\tau)$ *– intersects* L *in a point* $\boldsymbol{Q}(\tau)$ *for any* $\tau \in I$*, and the dependence on* τ *is* C^1.

5. *Fix* $\tau \in \mathbb{R}$*, and let* $\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\tau}}(d) \in W^u(\tau)$ *be such that* $\boldsymbol{\phi}(\tau;d) = \boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\tau}}(d)$*, for every* $d \geq 0$*. Then, the function* $\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\tau}} : [0,+\infty) \to W^u(\tau)$ *is a smooth (bijective) parametrization of* $W^u(\tau)$ *and* $\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\tau}}(0) = (0,0)$*. Analogously, the stable leave* $W^s(\tau)$ *can be parametrized directly by* $c := \lim_{r\to\infty} v(r) r^{n-2}$*. In particular, fix* $\tau \in \mathbb{R}$*, and let* $\boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\tau}}(c) \in W^s(\tau)$ *be such that* $\boldsymbol{\psi}(\tau;c) = \boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\tau}}(c)$*, for every* $c \geq 0$*. Then, the function* $\boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\tau}} : [0,+\infty) \to W^s(\tau)$ *is a smooth (bijective) parametrization of* $W^s(\tau)$ *and* $\boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\tau}}(0) = (0,0)$.

Since $\mathcal{K}$ is bounded, cf. (1.3), it is convenient to consider the autonomous system obtained from (2.2) by imposing $\mathcal{K}(t) \equiv 1 + c$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -(1+c)\, x^{q-1} \end{pmatrix}. \tag{2.4}$$

System (2.4) admits a unique critical point $\boldsymbol{P^*}(c) = (P^*(c), 0)$ in the halfplane $\{x > 0\}$. In particular, the value $P^*(c)$ is given by

$$P^*(c) = \left(\frac{\alpha^2}{c+1}\right)^{\frac{1}{q-2}}. \tag{2.5}$$

Proposition 2.3. *Assume* (1.3)*. Let* $\tau \in \mathbb{R}$ *and* $\boldsymbol{Q} = (X, 0)$ *with* $X > 0$ *and consider the trajectory* $\boldsymbol{\phi}(t; \tau, \boldsymbol{Q}) = (x(t; \tau, \boldsymbol{Q}), y(t; \tau, \boldsymbol{Q}))$ *of system* (2.2)*.*
Then, $y'(\tau; \tau, \boldsymbol{Q}) > 0$ *if* $X < P^*(\varepsilon) := \left(\frac{\alpha^2}{\varepsilon+1}\right)^{\frac{1}{q-2}}$.
Conversely, $y'(\tau; \tau, \boldsymbol{Q}) < 0$ *if* $X > P^*(0) := \left(\alpha^2\right)^{\frac{1}{q-2}}$.

We denote by $\boldsymbol{E_0}$ the set

$$\boldsymbol{E_0} = \left\{ (x, y) \mid \frac{y^2}{2} - \alpha^2 \frac{x^2}{2} + \frac{x^q}{q} \le 0 \,, \qquad x \ge 0 \right\} , \tag{2.6}$$

and by $\boldsymbol{\Gamma_0}$ its border. Let us briefly recall some contents of [16, §3].

Proposition 2.4. *Assume* (1.3) *and* $\mathbf{(H_1)}$*.*

1. *The energy function*

$$\mathcal{H}(x, y, t) := \frac{y^2}{2} - \alpha^2 \frac{x^2}{2} + \mathcal{K}(t) \frac{x^q}{q} \,, \tag{2.7}$$

   is decreasing along the trajectories of (2.2) *when* $t \le \mathcal{T} := \ln(\mathcal{R})$ *and increasing when* $t \ge \mathcal{T}$*. In fact, if* $\boldsymbol{\phi}(t) = (x(t), y(t))$ *solves* (2.2) *we have*

$$\frac{d\mathcal{H}(\boldsymbol{\phi}(t), t)}{dt} = \frac{d}{dt} [\mathcal{K}(t)] \, \frac{x^q(t)}{q} \,. \tag{2.8}$$

   Moreover, all the trajectories $\boldsymbol{\phi}(t; d) = (x(t; d), y(t; d))$ *and* $\boldsymbol{\psi}(s; c) = (\bar{x}(s; c), \bar{y}(s; c))$ *of system* (2.2) *belong to* $\boldsymbol{E_0}$ *for every* $t \le \mathcal{T} \le s$ *and* $d > 0$, $c > 0$*. In particular,* $x(t; d) > 0$ *for any* $t \le \mathcal{T}$ *and* $\bar{x}(t; c) > 0$ *for any* $t \ge \mathcal{T}$*.*

2. *Let* $\boldsymbol{\phi}(t) = (x(t), y(t))$ *be a trajectory of* (2.2)*; then the function*

$$H_c(x, y) := \frac{y^2}{2} - \alpha^2 \frac{x^2}{2} + (1 + c) \frac{x^q}{q} \tag{2.9}$$

   satisfies

$$\frac{d}{dt} H_c(x(t), y(t)) = \big[1 + c - \mathcal{K}(t)\big] y(t) \, x(t)^{q-1}. \tag{2.10}$$

   Hence, whenever $x(t) > 0$ *we find that* $H_c(x(t), y(t))$ *is increasing in* t *if* $c = \varepsilon$ *and* $y > 0$*, or if* $c = 0$ *and* $y < 0$*.*

From Proposition 2.4-1 combined with definition (2.3), we notice that, for every $\tau_1 \le \mathcal{T} \le \tau_2$, $W^u(\tau_1)$ and $W^s(\tau_2)$ are fully contained in the set $\boldsymbol{E_0}$ introduced in (2.6).

Proposition 2.5. *Assume* (1.3) *and* $\mathbf{(H_1)}$*. Then,*

$$W^u(\tau_1) \subset \boldsymbol{E_0}, \quad W^s(\tau_2) \subset \boldsymbol{E_0}, \qquad \forall \, \tau_1 \le \mathcal{T} \le \tau_2.$$

Let us mention a relevant consequence of Proposition 2.4-2.

Proposition 2.6. *Assume* (1.3) *and* $(\mathbf{H_1})$. *There exists a decreasing sequence of positive values* $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ *with the following property: for every* $0<\varepsilon<\varepsilon_\ell$ *we can construct a continuous and injective* spiral-like path $\boldsymbol{\gamma}$ *inside* $\boldsymbol{E_0}$ *which rotates around the points* $\boldsymbol{P^*}(\varepsilon)=(P^*(\varepsilon),0)$ *and* $\boldsymbol{P^*}(0)=(P^*(0),0)$ *for a total angle equal to* $2\pi\ell$, *cf. Figure 1.*
The spiral $\boldsymbol{\gamma}$ *is a* guiding curve *for system* (2.2) *in the following sense: the trajectories intersecting* $\boldsymbol{\gamma}$ *cross the path from the inner part to the outer part for any* $t\in\mathbb{R}$, *cf. Figure* 1.

The construction of the path $\boldsymbol{\gamma}$ is the object of [16, §3] and it is the tool that allows to establish a computable lower bound for the values η_ℓ and ε_ℓ. The explicit expression of the values ε_ℓ is given in Theorem 1.6 and in the table (1.6).

The *guiding curve* $\boldsymbol{\gamma}$ is obtained by appropriately gluing pieces of level curves of the functions H_ε and H_0 in the halfplanes $\{y\geq 0\}$ and $\{y\leq 0\}$, respectively.

Let us introduce some notation, cf. Figure 1. Let us follow clockwise the path $\boldsymbol{\gamma}$ from the origin onwards: we denote by $\boldsymbol{A_1}, \boldsymbol{A_2},\dots,\boldsymbol{A_{2\ell}}$ respectively the first, second, ... $2\ell^{th}$ (transversal) intersection of $\boldsymbol{\gamma}$ with the x positive semi-axis. We set $\boldsymbol{A_0}=(0,0)$ and we denote by $\mathcal{A}_i$ the branch of $\boldsymbol{\gamma}$ between $\boldsymbol{A_{i-1}}$ and $\boldsymbol{A_i}$, cf. [16, §3]. Moreover, setting $\boldsymbol{A_i}=(A_i,0)$, we find

$$0=A_0<A_2<\cdots<A_{2\ell}<A_{2\ell+2}<\cdots<P^*(\varepsilon)<P^*(0)< \\ <\cdots<A_{2\ell+3}<A_{2\ell+1}<\cdots<A_1\,. \quad (2.11)$$

We conclude the survey of known results by recalling a crucial asymptotic property of the trajectories $\boldsymbol{\phi}(t;d)$ of (2.2). The statement is a reformulation of the ones in [11, Theorem 1.6] and [10, Lemma 2.2].

Proposition 2.7. [16, Lemma 4.4] *Assume conditions* $(\mathbf{H_1})$-$(\mathbf{H_2^0})$ *and fix* $\tau\leq\mathcal{T}$. *There is an increasing sequence* $d_i\to+\infty$ *as* $i\to+\infty$ *satisfying the following property: for any* $i\in\mathbb{N}$ *the trajectory* $\boldsymbol{\phi}(\cdot;d_i)$ *is such that* $x(\cdot;d_i)>0$ *and* $y(\cdot;d_i)$ *has at least* $2i$ *non-degenerate zeroes in the time interval* $(-\infty,\tau)$.

Remark 2.8. *We emphasize that* $(\mathbf{H_2^0})$ *does not seem to be a technical requirement; in fact the existence of the sequence* d_i *in Proposition* 2.7 *is proved in [11, Theorem 1.6] as a consequence of the existence of a trajectory* $\boldsymbol{\phi}(t)$ *of* (2.2) *which has the whole* $\boldsymbol{\Gamma_0}$ *as* α*-limit set. However, in [11, Theorem 1.1] it is shown that such a trajectory cannot exist if* $l_0\geq\frac{n-2}{2}$ *in* $(\mathbf{H_2^0})$. *We conjecture that, if* $(\mathbf{H_2^0})$ *is removed, Theorem* 1.6 *does not hold and that there exists an upper bound* N, *with* $N\geq 1$, *of the number of GS with fast decay of* (1.8), *even for* $\varepsilon\to 0^+$.

According to Proposition 2.2-2, we need to locate the unstable leaves $W^u(\tau)$ in order to detect regular solutions of system (2.2), and we need to compute the number of laps performed by $W^u(\tau)$ around the points $\boldsymbol{P^*}(\varepsilon)$ and $\boldsymbol{P^*}(0)$ to ensure multiplicity. We are also interested in correlating the number of rotations drawn by the branch of $W^u(\tau)$ between the origin and its point $\boldsymbol{Q^u_\tau}(d)=\boldsymbol{\phi}(\tau;d)$, with the number of rotations realized by the solution $\boldsymbol{\phi}(\cdot;d)$ in $(-\infty,\tau]$.
In line with the behaviour of the trajectories of (2.2) illustrated in Propositions 2.4 and 2.6, fixed $\tau\leq\mathcal{T}$ we also expect that $W^u(\tau)$ rotates clockwise accompanied by the spiral $\boldsymbol{\gamma}$, and intersects 2ℓ-times the x axis, when $\varepsilon<\varepsilon_\ell$. This conjecture will be demonstrated in the next Section.

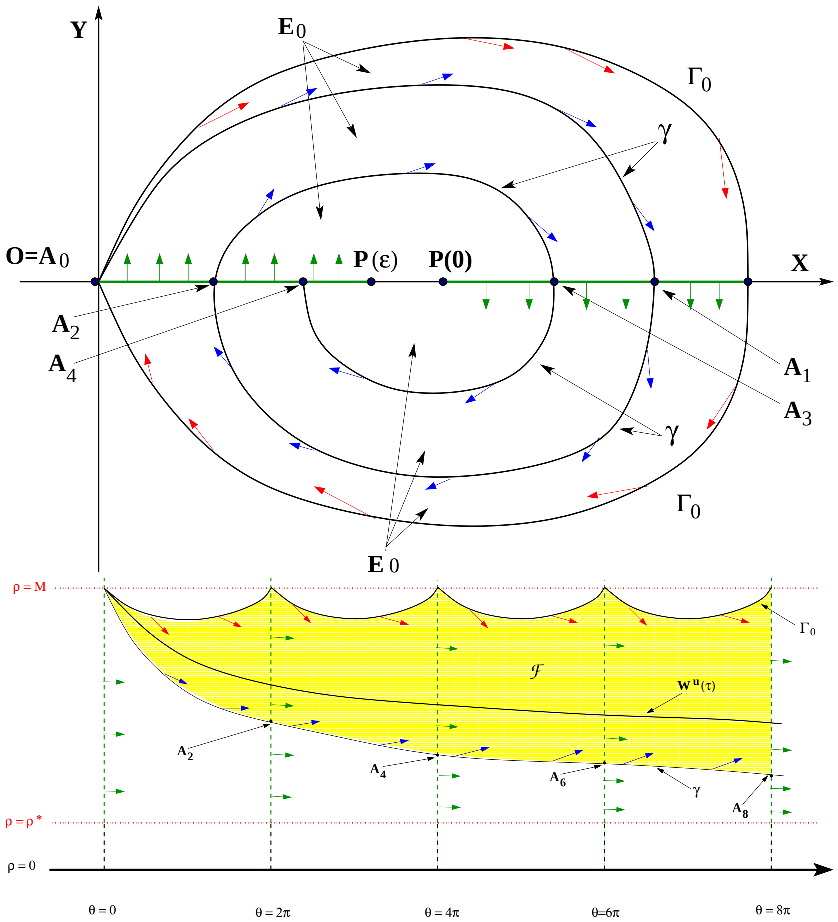

Figure 1: The path $\boldsymbol{\gamma}$ guides the trajectories of (2.2). The vector field (in blue) on $\boldsymbol{\gamma}$ and on the green part of the x axis is always transversal (Propositions 2.6 and 2.3), while on Γ_0 the vector field (in red) may admit some tangencies. However, Γ_0 cannot be crossed by $\boldsymbol{\phi}(t; d)$ for $t \leq \mathcal{T}$, see Proposition 2.4-1. Below, we give an interpetation in polar coordinates (3.1) of the picture above: a trajectory which is in $\mathcal{F}$ at the time τ is forced to remain inside $\mathcal{F}$, guided by $\boldsymbol{\gamma}$ and Γ_0, for every $t > \tau$ ($t \leq \mathcal{T}$) until it crosses the line $\theta = 8\pi$, provided that $0 < \varepsilon < \varepsilon_4$.

3 Properties of the unstable manifold

This Section is devoted to determine the position and the shape of $W^u(\tau)$ using polar coordinates. Let $\boldsymbol{M} = (M, 0)$ be the middle point between $\boldsymbol{P}^*(\varepsilon)$ and $\boldsymbol{P}^*(0)$, i.e. $\boldsymbol{M} = \frac{1}{2}(\boldsymbol{P}^*(\varepsilon) + \boldsymbol{P}^*(0))$. The polar coordinates with respect to $\boldsymbol{M}$ are given by:

$$\begin{cases} x = M - \rho \cos \theta \\ y = \rho \sin \theta \, . \end{cases} \tag{3.1}$$

Remark 3.1. *The path* $\boldsymbol{\gamma}$ *and the set* $\boldsymbol{\Gamma_0}$ *can be parametrized according to* (3.1). *In particular, for every* $0 < \varepsilon < \varepsilon_\ell$*, we define the function* $\gamma : [0, 2\pi\ell] \to \mathbb{R}^+$

such that

$$(x,y) = (M - \rho\cos\theta, \rho\sin\theta) \in \boldsymbol{\gamma} \iff \rho = \gamma(\theta).$$

Similarly, we define the function $\Gamma_0 : [0, 2\pi] \to \mathbb{R}^+$ *such that*

$$(x,y) = (M - \rho\cos\theta, \rho\sin\theta) \in \boldsymbol{\Gamma_0} \iff \rho = \Gamma_0(\theta).$$

Notice that Γ_0 *can be extended periodically, since* $\Gamma_0(0) = \Gamma_0(2\pi) = M$.
Observe that the spiral $\boldsymbol{\gamma}$*, the point* $\boldsymbol{M}$*, the functions* γ *and* Γ_0 *depend on* ε*, but we leave this dependence unsaid for simplicity.*

Finally, since $\boldsymbol{\gamma}$ *is a subset of the bounded region delimited by* $\boldsymbol{\Gamma_0}$*,* $\gamma(\theta) < \Gamma_0(\theta)$*, and from* (2.11)

$$0 < \gamma((2i+2)\pi) < \gamma(2i\pi) < \cdots < \gamma(2\pi) < \gamma(0) = M. \tag{3.2}$$

Moreover, by construction, $\gamma(\theta) > \rho^* := \frac{P^*(0) - P^*(\varepsilon)}{2} > 0$.

We now parametrize the trajectories $\boldsymbol{\phi}(t;d)$ of (2.2) corresponding to *regular solutions* $u(r;d)$ of (1.8) using the polar coordinates (3.1). We set

$$\boldsymbol{\phi}(t;d) = \big(M - \varrho_d^u(t)\cos(\vartheta_d^u(t))\,,\, \varrho_d^u(t)\sin(\vartheta_d^u(t))\big). \tag{3.3}$$

According to Proposition 2.2-2, the polar coordinates are well-defined in a neighborhood of $t = -\infty$: it is not restrictive to assume that

$$\lim_{t\to-\infty} \vartheta_d^u(t) = 0\,. \tag{3.4}$$

When no ambiguity arises, we drop the subscript in the polar coordinates (ϑ, ϱ).

Assume $0 < \varepsilon < \varepsilon_\ell$. From Propositions 2.5 and 2.6, we notice that the trajectory $\boldsymbol{\phi}(t;d)$, corresponding to a regular solution $u(r;d)$, is contained in $\boldsymbol{E_0}$ at least for any $t \le \mathcal{T}$ and it is guided by the spiral $\boldsymbol{\gamma}$ until it completes ℓ rotations, cf. Figure 1.
Let us set

$$\mathcal{F} = \{(\theta,\rho) \in \mathbb{R}^2 \mid \theta \in [0, 2\pi\ell]\,,\, 0 < \gamma(\theta) < \rho < \Gamma_0(\theta)\}\,. \tag{3.5}$$

We denote by $t_d \le \mathcal{T}$ the time at which $\boldsymbol{\phi}(t;d)$ leaves $\mathcal{F}$; if the trajectory remains in $\mathcal{F}$, we set $t_d := \mathcal{T}$. More precisely, we have the following.

Remark 3.2. *Assume* $\mathbf{(H_1)}$ *and* (1.3) *with* $0 < \varepsilon < \varepsilon_\ell$. *For every* $d > 0$, *there exists* $t_d \le \mathcal{T}$ *such that the polar coordinates in* (3.3) *are well-defined whenever* $t \in J_d = (-\infty, t_d)$ *and have the following property:*

$$\vartheta_d^u(t) \in (0, 2\pi\ell)\,, \quad (\vartheta_d^u(t), \varrho_d^u(t)) \in \mathcal{F}\,, \quad \textit{for every } t \in J_d\,. \tag{3.6}$$

Moreover, $\vartheta_d^u(t_d) \le 2\pi\ell$ *and if* $t_d < \mathcal{T}$ *then* $\vartheta_d^u(t_d) = 2\pi\ell$.

Hence, for any $d > 0$ we have

$$(\vartheta_d^u(t), \varrho_d^u(t)) \in \mathcal{F}\,, \quad \text{whenever } t \in J_d. \tag{3.7}$$

We are now able to restate Proposition 2.7 in terms of the angles.

Proposition 3.3. *Fix a positive integer ℓ. Assume conditions* $(\mathbf{H_1})$-$(\mathbf{H_2^0})$ *and* (1.3) *with $0<\varepsilon<\varepsilon_\ell$. Then, for every $\tau\le\mathcal{T}$ there is $D_\ell(\tau)>0$ such that*

$$\vartheta^u_{D_\ell(\tau)}(\tau)=2\ell\pi\,. \tag{3.8}$$

Without loss of generality, we assume that $D_\ell(\tau)$ is the smallest value which satisfies (3.8)*. Hence, $0<\vartheta^u_d(\mathcal{T})<2\ell\pi$ for any $0<d<D_\ell(\mathcal{T})$.*

The proof is obtained from a trivial adaptation of the proof provided in [16, Theorem 1.1], i.e. Theorem 1.2 of this article. Note that in [16, Theorem 1.1] the assumption $(\mathbf{H_1})$ holds under the restriction $\mathcal{R}=1$, or, equivalently, $\mathcal{T}=0$, and (3.8) is proved for $\tau=\mathcal{T}=0$. In fact, in [16] (cf. the conclusive sentence of the proof of Theorem A) the authors show the existence of a value $D_\ell(\mathcal{T})>0$ such that the trajectory $\boldsymbol{\phi}(t;D_\ell(\mathcal{T}))$ has the following properties: $x(t;D_\ell(\mathcal{T}))>0$ for any $t\le\mathcal{T}$ and $\vartheta^u_{D_\ell(\mathcal{T})}(\mathcal{T})=2\ell\pi$. Theorem 1.1 in [16] immediately follows from this property because of the even symmetry of $\mathcal{K}(t)$ (cf. assumption $(\mathbf{K_0})$), which in the present paper is not assumed. So, here we just need to repeat the proof in [16, Theorem 1.1] word by word, with $\mathcal{T}:=\ln(\mathcal{R})$ in place of 0 and a generic $\tau\le\mathcal{T}$.

However, we sketch the argument for completeness, remanding the interested reader to [16] for a full fledged proof.

Proof of Proposition 3.3. From Proposition 2.4-1, we have $\boldsymbol{\phi}(t;d)\in\boldsymbol{E_0}$ for any $t\le\mathcal{T}$ and any $d>0$. So, from Proposition 2.3 and Remark 3.2, see also Figure 1, we notice that the first 2ℓ possible intersections of $\boldsymbol{\phi}(\cdot;d)$ with the x positive semi-axis are trasversal in $(-\infty,\mathcal{T}]$. Then, fixed $\tau\le\mathcal{T}$, we define the set

$$I_{2\ell}(\tau):=\{d>0\mid y(t;d)\text{ has at least }2\ell\text{ zeroes for }t<\tau\}\,,$$

and we denote by $T_j(d)$ the j^{th} zero of $y(\cdot;d)$, for $j=1,\dots,2\ell$. Obviously, $T_{2\ell}(d)<\tau$ when $d\in I_{2\ell}(\tau)$. From Proposition 2.7 we deduce that there exists $d_\ell\in I_{2\ell}(\tau)$, so $I_{2\ell}(\tau)\neq\varnothing$. Furthermore, due to the transversality of the crossings, it can be shown that $I_{2\ell}(\tau)$ is open and that $T_{2\ell}(d)$ is continuous in $I_{2\ell}(\tau)$. According to [16, Remark 4.3], we also know that $d\notin I_{2\ell}(\tau)$ whenever $d>0$ is small enough; hence we can define

$$D_\ell(\tau):=\inf I_{2\ell}(\tau)>0. \tag{3.9}$$

Recalling Remark 3.2, if $d<D_\ell(\tau)$, then $t_d\ge\tau$ so that $(\vartheta^u_d(t),\varrho^u_d(t))\in\mathcal{F}$ for every $t\le\tau$. By continuity, Propositions 2.4-1 and 2.6 we conclude that $(\vartheta^u_{D_\ell(\tau)}(t),\varrho^u_{D_\ell(\tau)}(t))\in\mathcal{F}$ for every $t\le\tau$, in particular $\vartheta^u_{D_\ell(\tau)}(\tau)\le 2\pi\ell$. We cannot have $\vartheta^u_{D_\ell(\tau)}(\tau)<2\pi\ell$, otherwise by continuity we would find $\delta>0$ such that $\vartheta^u_d(\tau)<2\pi\ell$ and $(\vartheta^u_d(t),\varrho^u_d(t))\in\mathcal{F}$ for every $t\le\tau$, and any $d\in(D_\ell(\tau),D_\ell(\tau)+\delta)$, thus contradicting (3.9). Hence, $\vartheta^u_{D_\ell(\tau)}(\tau)=2\pi\ell$. □

As an immediate consequence, taking into account the definition of t_d given in Remark 3.2, we observe that

$$t_{D_\ell(\tau)}=\tau\quad\text{ and }\quad\vartheta^u_d(\tau)\in(0,2\pi\ell)\quad\text{for }0<d<D_\ell(\tau)\,,\ \tau\le\mathcal{T}\,. \tag{3.10}$$

We now focus on the leaf $W^u(\mathcal{T})$: we recall it is a set of initial conditions converging to the origin, and a priori it is not a graph of a trajectory unless the system is autonomous. Moreover, the transversality of its intersections with the x positive semi-axis is not guaranteed.

Taking into account Proposition 2.2-5, we can introduce the parametrization in polar coordinates of $W^u(\mathcal{T})$, by setting

$$\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\mathcal{T}}}(d) = \boldsymbol{\phi}(\mathcal{T}; d) := \big(M - \mathscr{R}^u_{\mathcal{T}}(d) \cos(\xi^u_{\mathcal{T}}(d)) \, , \mathscr{R}^u_{\mathcal{T}}(d) \sin(\xi^u_{\mathcal{T}}(d)) \big) \, , \qquad (3.11)$$

where $d > 0$ is small enough. Recalling that $\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\mathcal{T}}}(0) = (0, 0)$, we may assume that

$$\xi^u_{\mathcal{T}}(0) = 0 \, . \qquad (3.12)$$

We are now interested in correlating the number of rotations of $W^u(\mathcal{T})$ with the ones of the regular solutions of (2.2). To this aim, we follow the techniques introduced and developed in the papers [30] and [2], dealing with autonomous problems. For extensions to non-autonomous settings, we refer to [15, 18, 23, 25, 35].

From Remark 3.2 and Proposition 3.3, if $0 < \varepsilon < \varepsilon_\ell$ the polar coordinates (3.11) are well defined in the set $[0, D_\ell(\mathcal{T})]$, since $\varrho^u_d(\mathcal{T}) > 0$, cf. (3.5). We now show that the number of rotations around $\boldsymbol{M}$ realized by the flow $\boldsymbol{\phi}(\cdot; d)$ in $(-\infty, \mathcal{T}]$ equals the number of rotations performed by the branch of $W^u(\mathcal{T})$ between the origin and its point $\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\mathcal{T}}}(d) = \boldsymbol{\phi}(\mathcal{T}; d)$.

Lemma 3.4. *Fix a positive integer ℓ. Assume conditions* $\mathbf{(H_1)}$-$\mathbf{(H_2^0)}$ *and* (1.3) *with $0 < \varepsilon < \varepsilon_\ell$. For any $d \in [0, D_\ell(\mathcal{T})]$, consider the point $\boldsymbol{Q} = \boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\mathcal{T}}}(d) \in W^u(\mathcal{T})$. Then,*

$$\vartheta^u_d(\mathcal{T}) = \xi^u_{\mathcal{T}}(d), \qquad (3.13)$$

i.e. the angle $\xi^u_{\mathcal{T}}(d)$, performed by the branch of $W^u(\mathcal{T})$ between the origin and $\boldsymbol{Q}$ in a rotation around $\boldsymbol{M}$, equals the one performed by the trajectory $\boldsymbol{\phi}(\cdot; d) = \boldsymbol{\phi}(\cdot; \mathcal{T}, \boldsymbol{Q})$ in the interval of time $(-\infty, \mathcal{T}]$.

Proof. The proof is achieved by exhibiting an homotopy between the two curves and it is given in [15, Lemmas 3.3, 3.5] which is a slight variant of the argument in [2], see also [25, Lemma 4.3] for a detailed proof in a more general context. □

Remark 3.5. *From Proposition 3.3 and Lemma 3.4, we deduce that the branch of $W^u(\mathcal{T})$ between the origin and $\boldsymbol{Q}^{\boldsymbol{u}}_{\boldsymbol{\mathcal{T}}}(D_\ell(\mathcal{T}))$ performs exactly ℓ complete rotations around $\boldsymbol{M}$.*

Remark 3.6. *From Remark 3.2,* (3.10) *and Lemma 3.4, we see that*

$$\xi^u_{\mathcal{T}}(d) \in (0, 2\pi\ell) \, , \quad (\xi^u_{\mathcal{T}}(d), \mathscr{R}^u_{\mathcal{T}}(d)) \in \mathcal{F} \, , \qquad \textit{for any } 0 < d < D_\ell(\mathcal{T}) \, , \qquad (3.14)$$

and $\xi^u_{\mathcal{T}}(D_\ell(\mathcal{T})) = 2\pi\ell$, $\rho^* < \mathscr{R}^u_{\mathcal{T}}(D_\ell(\mathcal{T})) < M$, where $\rho^* = \frac{P^*(0) - P^*(\varepsilon)}{2} > 0$ is defined in Remark 3.1.

Remark 3.7. *It is easy to check that equations* (3.11)*,* (3.12)*, Lemma 3.4 and Remarks 3.5, 3.6 hold, in fact, for any $\tau \le \mathcal{T}$ in the place of $\mathcal{T}$. If we choose $\tau > \mathcal{T}$ we cannot ensure anymore that the trajectories do not touch $\boldsymbol{M}$, so our argument fails.*

The existence of GS with fast decay corresponds to the existence of intersections between $W^u(\mathcal{T})$ and $W^s(\mathcal{T})$. For this reason, we now concentrate on the study of the stable leaves.

4 Kelvin inversion and stable manifold

An important tool in the analysis of equation (1.2) is a change of variables classically known as Kelvin inversion, useful to convert the information from the unstable leaves to the stable ones. Set

$$s = r^{-1}, \quad \tilde{u}(s) = s^{2-n} u(1/s), \quad \tilde{K}(s) = K(1/s). \tag{4.1}$$

From a straightforward computation, $u(r)$ satisfies (1.2) if and only if $\tilde{u}(s)$ satisfies the following equation

$$\frac{d}{ds}\left[\frac{d\tilde{u}}{ds}\, s^{n-1}\right] + \tilde{K}(s)\, s^{n-1}\, \tilde{u}^{q-1} = 0. \tag{4.2}$$

Thus, the change of variables (4.1) brings regular solutions of (1.2) into fast decay solutions $\tilde{u}(s)$ of (4.2), and viceversa.
If K satisfies $(\mathbf{H_1})$, then $\tilde{K}$ satisfies the equivalent condition $(\widetilde{\mathbf{H}}_{\mathbf{1}})$:

$(\widetilde{\mathbf{H}}_{\mathbf{1}})$ $\frac{d\tilde{K}}{ds} \le 0$ and $\frac{d\tilde{K}}{ds} \not\equiv 0$ in $(0, 1/\mathcal{R})$,
$\frac{d\tilde{K}}{ds} \ge 0$ and $\frac{d\tilde{K}}{ds} \not\equiv 0$ in $(1/\mathcal{R}, +\infty)$.

Moreover, condition $(\mathbf{H_2^\infty})$ on K corresponds to the following condition on $\tilde{K}$:

$(\widetilde{\mathbf{H}}_{\mathbf{2}})$ $\tilde{K}(s) = K_\infty - a_\infty s^{l_\infty} + \tilde{h}_\infty(s)$, where $\tilde{h}(s) = h(1/s)$,

$$a_\infty > 0,\ 0 < l_\infty < \frac{n-2}{2},\ \lim_{s\to 0} \left|\tilde{h}_\infty(s)\right| s^{-l_\infty} + \left|\frac{d\tilde{h}_\infty(s)}{ds}\right| s^{-l_\infty+1} = 0.$$

Note that assumption $(\widetilde{\mathbf{H}}_{\mathbf{2}})$ is equivalent to assumption $(\mathbf{H_2^0})$, provided that we replace a_0 and l_0 respectively with a_∞ and l_∞ in $(\widetilde{\mathbf{H}}_{\mathbf{2}})$. According to [23], we combine the Kelvin inversion with the Fowler transformation (2.1) by setting

$$s = \mathrm{e}^T, \quad \tilde{x}(T) = \tilde{u}(s)s^\alpha, \quad \tilde{y}(T) = \alpha\tilde{u}(s)s^\alpha + \frac{d\tilde{u}(s)}{ds}s^{\alpha+1}. \tag{4.3}$$

The change of variables (4.3) transforms (4.1) into the equivalent two-dimensional dynamical system, see [23, p. 521] for a full fledged argument:

$$\begin{pmatrix} \frac{d\tilde{x}}{dT} \\ \frac{d\tilde{y}}{dT} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} 0 \\ -\widetilde{\mathcal{K}}(T)\, \tilde{x}^{q-1} \end{pmatrix}, \tag{4.4}$$

where $\widetilde{\mathcal{K}}(T) = \tilde{K}(e^T) = \mathcal{K}(-T)$. Due to the reciprocal analogy, the study of system (4.4) reduces to the study of system (2.2). In particular, system (4.4) admits unstable and stable leaves, $\widetilde{W}^u(-\tau)$ and $\widetilde{W}^s(-\tau)$, which satisfy the same properties of $W^u(\tau)$ and $W^s(\tau)$, respectively.

Notice that the trajectory $\boldsymbol{\phi}(t) = (x(t), y(t))$ solves system (2.2) if and only if $\tilde{\boldsymbol{\phi}}(T) := (\tilde{x}(T), \tilde{y}(T))$ solves (4.4), where

$$T = -t, \qquad \tilde{x}(T) = x(t), \qquad \tilde{y}(T) = -y(t), \tag{4.5}$$

and, obviously, $\widetilde{\mathcal{K}}(T) = \mathcal{K}(t)$. As an immediate consequence, we get the following property.

Remark 4.1. *The stable leaves $W^s(\tau)$ of* (2.2) *and the unstable leaves $\widetilde{W}^u(-\tau)$ of* (4.4) *are symmetric with respect to the x axis, as well as $W^u(\tau)$ and $\widetilde{W}^s(-\tau)$.*

This enables us to automatically reduce the study of the stable leaves $W^s(\tau)$ of (2.2) to the study of the unstable leaves $\widetilde{W}^u(-\tau)$ of (4.4).
Hence, each property of the regular solutions of (2.2) and of the unstable leaves described in the previous sections can be translated into that of fast decay solutions and of stable leaves, respectively.

Fix $\varepsilon \in (0, \varepsilon_\ell)$. Let $\widetilde{\boldsymbol{\gamma}}$ be the spiral obtained from $\boldsymbol{\gamma}$ with a reflection with respect to the x axis, i.e.

$$\widetilde{\boldsymbol{\gamma}} := \{(M - \rho\cos\theta, -\rho\sin\theta) \mid \rho = \gamma(\theta), \quad 0 \le \theta \le 2\ell\pi\}.$$

Equivalently, let us define $\widetilde{\gamma} : [-2\ell\pi, 0] \to \mathbb{R}^+$, by setting $\widetilde{\gamma}(\theta) = \gamma(-\theta)$ for any $-2\ell\pi \le \theta \le 0$. Thus, we obtain

$$(x, y) = (M - \rho\cos\theta, \rho\sin\theta) \in \widetilde{\boldsymbol{\gamma}} \iff \rho = \widetilde{\gamma}(\theta) = \gamma(-\theta).$$

Similarly, let $\widetilde{\mathcal{F}}$ be the set obtained from $\mathcal{F}$ with a reflection with respect to the x axis, i.e.

$$\widetilde{\mathcal{F}} := \{(\theta, \rho) \in \mathbb{R}^2 \mid \theta \in [-2\pi\ell, 0], \widetilde{\gamma}(\theta) < \rho < \Gamma_0(\theta)\}.$$

Let us focus on the stable leaf $W^s(\mathcal{T})$. Taking into account Proposition 2.2-5, we parametrize $W^s(\mathcal{T})$ in polar coordinates, by setting

$$\boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\mathcal{T}}}(c) = \boldsymbol{\psi}(\mathcal{T}; c) := (\, M - \mathscr{R}^s_{\mathcal{T}}(c)\, \cos(\xi^s_{\mathcal{T}}(c)) \,, \mathscr{R}^s_{\mathcal{T}}(c)\, \sin(\xi^s_{\mathcal{T}}(c))\,)\,, \tag{4.6}$$

for every $c \ge 0$. Recalling that $\boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\mathcal{T}}}(0) = (0, 0)$, we may assume that

$$\xi^s_{\mathcal{T}}(0) = 0\,. \tag{4.7}$$

From Remarks 3.6 and 4.1, we easily obtain the following translation of Remark 3.6.

Proposition 4.2. *Fix a positive integer ℓ. Assume conditions* $(\mathbf{H_1})$, $(\mathbf{H_2^\infty})$ *and* (1.3) *with $0 < \varepsilon < \varepsilon_\ell$. Consider the parametrization $\boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\mathcal{T}}}$ of the stable manifold $W^s(\mathcal{T})$, as in* (4.6)*. Then, there is $c_\ell(\mathcal{T})$ such that*

$$\xi^s_{\mathcal{T}}(c) \in (-2\pi\ell, 0), \qquad (\xi^s_{\mathcal{T}}(c), \mathscr{R}^s_{\mathcal{T}}(c)) \in \widetilde{\mathcal{F}}, \qquad \text{for every } c \in (0, c_\ell(\mathcal{T})),$$

and

$$\xi^s_{\mathcal{T}}(c_\ell(\mathcal{T})) = -2\ell\pi\,, \qquad \rho^* < \mathscr{R}^s_{\mathcal{T}}(c_\ell(\mathcal{T})) < M\,.$$

where $\rho^* = \frac{P^*(0) - P^*(\varepsilon)}{2} > 0$ is defined in Remark 3.1.

The parametrization of the trajectories $\boldsymbol{\psi}(t; c)$ of (2.2) corresponding to *fast decay solutions* $v(r; c)$ of (1.8) is given by

$$\boldsymbol{\psi}(t; c) = \big(M - \varrho^s_c(t)\cos(\vartheta^s_c(t))\,,\, \varrho^s_c(t)\sin(\vartheta^s_c(t))\big), \tag{4.8}$$

where, according to Proposition 2.2-3, it is not restrictive to set $\lim_{t\to+\infty} \vartheta^s_c(t) = 0$.

The next lemma is the translation of Lemma 3.4 for the stable manifold.

Lemma 4.3. *Fix a positive integer ℓ. Assume conditions* $(\mathbf{H_1})$, $(\mathbf{H_2^\infty})$ *and* (1.3) *with $0 < \varepsilon < \varepsilon_\ell$. For any $c \in [0, c_\ell(\mathcal{T})]$, consider the point $\boldsymbol{Q}^{\boldsymbol{s}} = \boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\mathcal{T}}}(c) \in W^s(\mathcal{T})$. Then,*

$$\vartheta^s_c(\mathcal{T}) = \xi^s_{\mathcal{T}}(c) < 0, \tag{4.9}$$

i.e. the trajectory $\boldsymbol{\psi}(\cdot; c) = \boldsymbol{\phi}(\cdot; \mathcal{T}, \boldsymbol{Q}^{\boldsymbol{s}}_{\boldsymbol{\mathcal{T}}})$ performs in the interval of time $[\mathcal{T}, +\infty)$ exactly the angle $-\xi^s_{\mathcal{T}}(c)$ around $\boldsymbol{M}$.

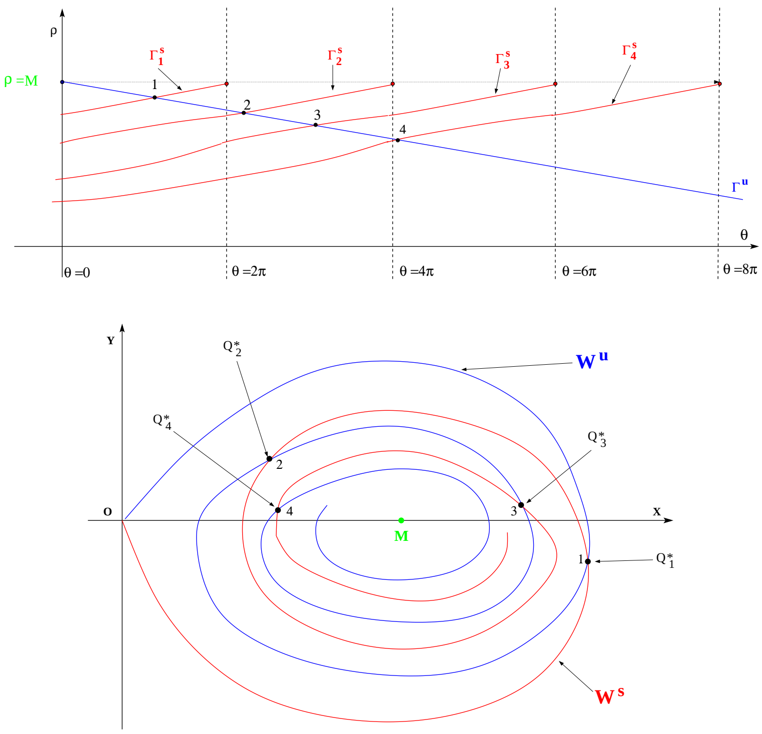

Figure 2: Intersections between the (blue) curve $\widetilde{\Gamma}^u$ and the (red) curves Γ_i^s for $i = 1, 2, 3, 4$, in the (θ, ρ) plane (above), and the corresponding intersections in the (x, y) plane (below) between $W^u(\mathcal{T})$ (blue) and $W^s(\mathcal{T})$ (red). Here we illustrate the easiest case where the intersection between $\widetilde{\Gamma}^u$ and Γ_i^s is unique for $i = 1, 2, 3, 4$, so there are 4 GS with fast decay. The trajectory $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q_i^*})$ completes exactly i loops around $\boldsymbol{M}$, for $i = 1, 2, 3, 4$.

5 Main result

This section is devoted to detect intersections between $W^u(\mathcal{T})$ and $W^s(\mathcal{T})$, which correspond to ground states of (1.8) having fast decay.

Lemma 5.1. *Fix a positive integer ℓ. Assume conditions* $(\mathbf{H_1})$-$(\mathbf{H_2^0})$-$(\mathbf{H_2^\infty})$ *and* (1.3) *with $0 < \varepsilon < \varepsilon_\ell$. Then $W^u(\mathcal{T})$ intersects $W^s(\mathcal{T})$ at least in ℓ points $\boldsymbol{Q_j^*}$, with $j \in \{1, \dots, \ell\}$.*

Proof. The proof is based on the same ideas developed in the proof of [15, Lemma 3.9]. Let us fix $\varepsilon \in (0, \varepsilon_\ell)$.

According to the parametrizations (3.11) of $W^u(\mathcal{T})$ and (4.6) of $W^s(\mathcal{T})$, we define the curves $\Gamma^u : [0, +\infty) \to \mathbb{R} \times [0, +\infty)$ and $\Gamma^s : [0, +\infty) \to \mathbb{R} \times [0, +\infty)$ by setting

$$\Gamma^u(d) := (\xi^u_{\mathcal{T}}(d), \mathscr{R}^u_{\mathcal{T}}(d)) \quad \text{and} \quad \Gamma^s(c) := (\xi^s_{\mathcal{T}}(c), \mathscr{R}^s_{\mathcal{T}}(c))\,. \tag{5.1}$$

Note that Γ^u and Γ^s are the liftings of $W^u(\mathcal{T})$ and $W^s(\mathcal{T})$, respectively. Since $\tau = \mathcal{T}$ is fixed, from now to the end of the proof we omit the dependence of

$\xi^u_{\mathcal{T}}(d)$, $\mathscr{R}^u_{\mathcal{T}}(d)$, $\xi^s_{\mathcal{T}}(c)$, $\mathscr{R}^s_{\mathcal{T}}(c)$, $D_\ell(\mathcal{T})$, $c_\ell(\mathcal{T})$, ... on this variable to deal with less cumbersome notation.
Remark 3.6 ensures the existence of $D_\ell > 0$ such that $\xi^u(D_\ell) = 2\pi\ell$. Thus, recalling that $\Gamma^u(0) = (0, M)$, from the continuity of ξ^u we deduce that for every $j \in \{1, \ldots, \ell\}$ there exists $\tilde{d}_j \in (0, D_\ell]$ satisfying

$$\xi^u(\tilde{d}_j) = 2\pi j \quad \text{and} \quad 0 < \xi^u(d) < 2\pi j \quad \forall\, d \in (0, \tilde{d}_j).$$

Observe that by construction the sequence $\tilde{d}_j$ is increasing, and we have $\tilde{d}_\ell = D_\ell$. From Proposition 2.5, we deduce that $\mathscr{R}^u(\tilde{d}_j) < M$ for any $j \in \{1, \ldots, \ell\}$. Furthermore, since $W^u(\mathcal{T})$ cannot have self-intersections, we find that

$$0 < \mathscr{R}^u(\tilde{d}_\ell) < \mathscr{R}^u(\tilde{d}_{\ell-1}) < \ldots < \mathscr{R}^u(\tilde{d}_1) < M\,. \tag{5.2}$$

We denote by $\widetilde{\Gamma}^u$ the restriction of Γ^u to the interval $[0, \tilde{d}_\ell]$, i.e.

$$\widetilde{\Gamma}^u := \Gamma^u_{|[0,\tilde{d}_\ell]} : [0, \tilde{d}_\ell] \to [0, 2\pi\ell] \times [0, +\infty).$$

Analogous arguments apply to the curve Γ^s defined in (5.1). In particular, by Proposition 4.2 there exists $c_\ell > 0$ such that $\xi^s(c_\ell) = -2\pi\ell$. Thus, recalling the equality $\Gamma^s(0) = (0, M)$, for every $j \in \{1, \ldots, \ell\}$ we deduce the existence of $\tilde{c}_j \in (0, c_\ell]$ satisfying

$$\xi^s(\tilde{c}_j) = -2\pi j \quad \text{and} \quad -2\pi j < \xi^s(c) < 0 \quad \forall\, c \in (0, \tilde{c}_j).$$

Again by construction, the sequence $\tilde{c}_j$ is increasing and $\tilde{c}_\ell = c_\ell$. We now consider the following translations in the angular-coordinate of the curve Γ^s

$$\Gamma^s_j(c) := (\xi^s(c) + 2\pi j, \mathscr{R}^s(c))\,, \tag{5.3}$$

for every $j \in \{0, 1, \ldots, \ell\}$. Note that the curve Γ^s_j cannot intersect Γ^s_k if $j \neq k$, since $W^s(\mathcal{T})$ cannot have self-intersections.
Moreover, observe that $\Gamma^s_0 = \Gamma^s$, $\Gamma^s_j(0) = (2\pi j, M)$ and $\Gamma^s_j(\tilde{c}_j) = (0, \mathscr{R}^s(\tilde{c}_j))$. Using also Proposition 2.5, we find

$$0 < \mathscr{R}^s(\tilde{c}_\ell) < \mathscr{R}^s(\tilde{c}_{\ell-1}) < \ldots < \mathscr{R}^s(\tilde{c}_1) < M\,. \tag{5.4}$$

Summarizing, we have $\widetilde{\Gamma}^u(0) = (0, M)$, and, for any $j \in \{1, \ldots, \ell\}$, $\Gamma^s_j(\tilde{c}_j) = (0, \mathscr{R}^s(\tilde{c}_j))$ with $\mathscr{R}^s(\tilde{c}_j) < M$, and $\Gamma^s_j(0) = (2\pi j, M)$, $\widetilde{\Gamma}^u(\tilde{d}_j) = (2\pi j, \mathscr{R}^u(\tilde{d}_j))$ with $\mathscr{R}^u(\tilde{d}_j) < M$. Thus, from a continuity argument, it follows that for any fixed $j \in \{1, \ldots, \ell\}$, the graphs of Γ^s_j and $\widetilde{\Gamma}^u$ intersect at least in a point, say $\widetilde{\Gamma}^u(d^*_j)$, where $0 < d^*_j < \tilde{d}_j$, see Figures 2 and 3. We emphasize that

$$\boldsymbol{Q^*_j} := \big(M - \mathscr{R}^u(d^*_j)\cos(\xi^u(d^*_j))\,, \mathscr{R}^u(d^*_j)\sin(\xi^u(d^*_j))\big) \in W^u(\mathcal{T}) \cap W^s(\mathcal{T})$$

for any $j \in \{1, \ldots, \ell\}$.
Note that $\boldsymbol{Q^*_j} \neq \boldsymbol{Q^*_k}$ for $j \neq k$, since Γ^s_j and Γ^s_k cannot intersect; so we have found ℓ distinct points in $(W^u(\mathcal{T}) \cap W^s(\mathcal{T}))$ and the Lemma is proved. □

Let $c^*_j \in (0, \tilde{c}_j)$ be the values such that $\widetilde{\Gamma}^u(d^*_j) = \Gamma^s_j(c^*_j)$ so that

$$\big(M - \mathscr{R}^s(c^*_j)\cos(\xi^s(c^*_j))\,, \mathscr{R}^s(c^*_j)\sin(\xi^s(c^*_j))\big) = \boldsymbol{Q^*_j}.$$

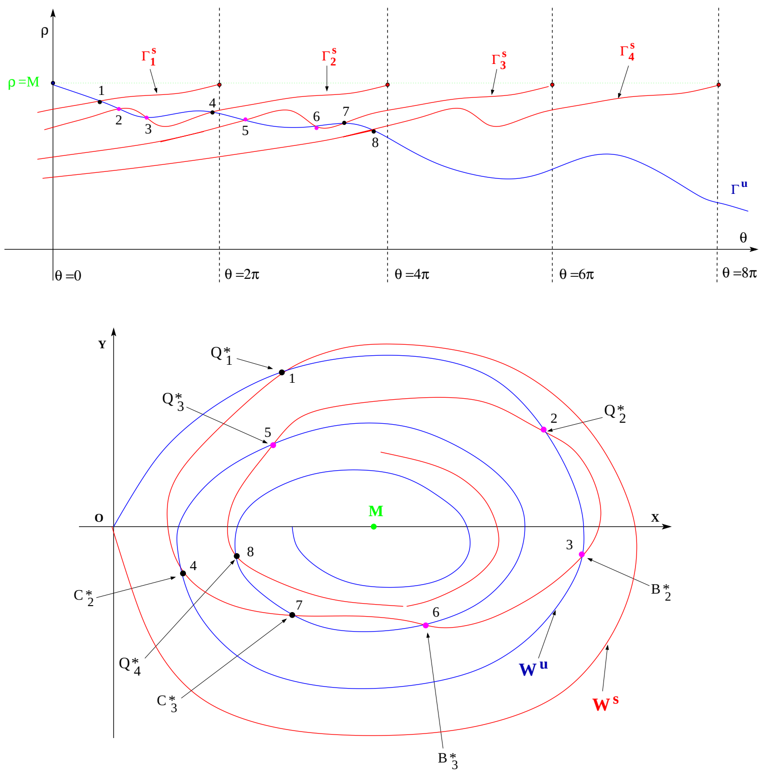

Figure 3: Intersections between the (blue) curve $\widetilde{\Gamma}^u$ and the (red) curves Γ_i^s for $i = 1, 2, 3, 4$, in the (θ, ρ) plane (above) and the corresponding intersections in the (x, y) plane between $W^u(\mathcal{T})$ and $W^s(\mathcal{T})$ (below). Here we illustrate the more difficult case where the intersection between $\widetilde{\Gamma}^u$ (in blue) and Γ_i^s (in red) is not unique for $i = 2, 3$ and there are 8 GS with fast decay. The trajectories $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q_j^*})$, $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{B_h^*})$, $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{C_h^*})$ complete exactly j and h loops around $\boldsymbol{M}$, for $j = 1, 2, 3, 4$, and $h = 2, 3$.

According to (5.1) and (5.3), we infer that

$$\xi_{\mathcal{T}}^u(d_j^*) = \xi_{\mathcal{T}}^s(c_j^*) + 2\pi j\,. \tag{5.5}$$

We are now ready to prove our main result Corollary 1.7 from which Theorem 1.6 follows as a special case.

Proof of Corollary 1.7. By construction, $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q_j^*})$ is a homoclinic trajectory of (2.2), and the corresponding solution $u_j(r) := u(r; d_j^*)$ of (1.8) is regular and has fast decay.

Furthermore, we have $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q_j^*}) \in \boldsymbol{E_0}$ for any $t \in \mathbb{R}$. In fact, $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q_j^*}) \in W^u(t) \subset \boldsymbol{E_0}$ for any $t \leq \mathcal{T}$ and $\boldsymbol{\phi}(t; \mathcal{T}, \boldsymbol{Q_j^*}) \in W^s(t) \subset \boldsymbol{E_0}$ for any $t \geq \mathcal{T}$, see Proposition 2.5.

Hence, $x(t; \mathcal{T}, \boldsymbol{Q_j^*}) > 0$ for any $t \in \mathbb{R}$, whence $u(r; d_j^*)$ is positive for any $r > 0$, and, consequently, it is a GS with fast decay.

Moreover, from Lemma 3.4 and Lemma 4.3, we know that $\boldsymbol{\phi}(\cdot; \mathcal{T}, \boldsymbol{Q_j^*})$ performs the angle $\vartheta^u_{d_j^*}(\mathcal{T}) = \xi^u_{\mathcal{T}}(d_j^*)$ around $\boldsymbol{M}$ in $(-\infty, \mathcal{T}]$, and the angle $-\xi^s_{\mathcal{T}}(c_j^*)$

in $[\mathcal{T},+\infty)$. Therefore, relation (5.5) ensures that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q_j^*})$ performs for $t\in\mathbb{R}$ the angle

$$\xi^u_{\mathcal{T}}(d_j^*)-\xi^s_{\mathcal{T}}(c_j^*)=2\pi j\,.$$

This implies that $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q_j^*})$ for $t\in\mathbb{R}$ makes exactly j rotations clockwise around $\boldsymbol{M}$. Furthermore, the crossings of $\boldsymbol{\phi}(t;\mathcal{T},\boldsymbol{Q_j^*})$ with the x positive semi-axis are all transversal, so $x(t;\mathcal{T},\boldsymbol{Q_j^*})$ has exactly j local maxima on the right side of $\boldsymbol{M}$ and $(j-1)$ local minima on the left side. The thesis immediately follows from transformation (2.1). □

Remark 5.2. *We can assume w.l.o.g. that the sequence d_j^* is increasing, simply redefining d_j^* as follows*

$$d_j^*:=\min\{D\in(0,\tilde{d}_j):\widetilde{\Gamma}^u(D)\in\Gamma^s_j((0,\tilde{c}_\ell))\}\,.$$

In fact, since the curve Γ^s_j cannot intersect Γ^s_k if $j\neq k$, the monotonicity of d_j^* easily follows. We refer to [15, Remark 3.10] for more details.

Remark 5.3. *We emphasize that, a priori, the curves Γ^s_j and $\widetilde{\Gamma}^u$ may have several intersections: in this case equation* (1.8) *admits more than ℓ GS with fast decay, see Fig. 3.*

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